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On the dynamics of a non-linear Duopoly game model

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ABSTRACT

The Cournot duopoly game modeled by Kopel, with adaptive expectations, is generalized by introducing the self-diffusion and cross-diffusion terms. General properties, such as boundedness and uniqueness, are obtained. Non-linear stability results are reached by the analysis of the stability of a ODE system.

1. Introduction

Oligopoly is the case where the market is controlled by a few number of firms producing similar products. In this paper we will restrict ourselves to the case of two firms, which is called duopoly. The situation in a duopoly is complex, since both firms have to take into account not only the behaviours of the consumers, but actions and reactions of the competitor. The first well-known model which gives a mathematical description of competition in a duopoly market dates back to the French economist Cournot (1838). In this paper, our starting point is the general case of the Cournot duopoly problem, which was modeled by Kopel (see [1–3]); precisely, we will examine the continuous time-scale counterpart of the above-mentioned non-linear Cournot–Kopel duopoly game.

Assuming continuous time scales, denoting by u and v the outputs of the two firms X and Y, respectively, a non-linear dynamic system for the evolution of u and v is obtained:

$$\begin{cases} \partial_t u = -\alpha_1 u + \alpha_1\mu_1 v(1-v) \\ \partial_t v = -\alpha_2 v + \alpha_2\mu_2 u(1-u) \end{cases} \tag{1}$$

where, in general, μ_i and α_i with $(i=1,2)$ are positive model parameters, ∂_t denotes the derivative with respect to time, $u:(\mathbf{x},t)\in\Omega\times\mathbb{R}^+\to u(\mathbf{x},t)\in\mathbb{R}$, $v:(\mathbf{x},t)\in\Omega\times\mathbb{R}^+\to v(\mathbf{x},t)\in\mathbb{R}$, Ω being a bounded domain in $\mathbb{R}^3$ with smooth boundary $\partial\Omega$.

To be realistic, any dynamical economic model should take into account both the time evolution and the spatial dependence of the characteristic variables.

In this paper, in order to generalize the model (1), we take into account the spatial dependence, by introducing the self-diffusion and cross-diffusion terms, by considering the following equations:

$$\begin{cases} \partial_t u = -\alpha_1 u + \alpha_1\mu_1 v(1-v) + \gamma_{11}\Delta u + \gamma_{12}\Delta v \\ \partial_t v = -\alpha_2 v + \alpha_2\mu_2 u(1-u) + \gamma_{21}\Delta u + \gamma_{22}\Delta v \end{cases} \tag{2}$$

where Δ denotes the Laplacian operator, $\gamma_{ij}=$ constant for $(i,j=1,2)$ and

$$\sum_{i,j=1}^{2}\gamma_{ij}\xi_i\xi_j \ge k|\xi|^2 \quad k>0,\ \ \xi=(\xi_1,\xi_2). \tag{3}$$

Already in [4,5] the authors introduce a new class of economic dynamical models, which are called "morphogenetic systems", which are constructed in order to generalize classical Goodwin's model of business cycle. Non-linear reaction–diffusion equations and systems play an important role in the modeling and study of many phenomena (see, for instance, dynamics of competing species, chemical aggression and convection problems in porous media, non-linear heat conduction, semiconductor devices, in [6–26]).

To (2) we append the initial data

$$u(\mathbf{x},0)=u_0(\mathbf{x}),\quad v(\mathbf{x},0)=v_0(\mathbf{x}) \text{ for } \mathbf{x}\in\Omega \tag{4}$$

and the following boundary conditions:

Dirichlet boundary conditions

$$u=\overline{u} \quad \text{and } v=\overline{v} \text{ on } \partial\Omega\times\mathbb{R}^+, \tag{5}$$

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where $\overline{u}$, $\overline{v}$ will be chosen among the constant steady-states solutions of (2), or Robin boundary data

$$\begin{cases} \beta u+(1-\beta)\nabla u\cdot\mathbf{n}=\overline{u}\beta \\ \beta v+(1-\beta)\nabla v\cdot\mathbf{n}=\overline{u}\beta, \end{cases} \quad \text{on } \partial\Omega\times\mathbb{R}^+ \tag{6}$$

where $0<\beta<1$.

The aim of the present paper is to analyse the non-linear L^2-stability of the constant steady states of (2) by following the procedure introduced by Rionero in [6] and used in [7–11]. Precisely, our aim is to link the stability (instability) of the generic equilibrium to (2) to the stability (instability) of the null solution to a linear system of ordinary differential equations associated to (2). The plan of the paper is the following. Section 2 is dedicated to the derivation of the mathematical model at hand, while, in Section 3, we consider the perturbation equations associated to the generalized model. Section 4 concerns boundedness and uniqueness of perturbations, while Section 5 is devoted to recall some results which allow to obtain conditions guaranteeing the stability (Section 6) of a critical point. Section 7 concerns a simplified case in which the obtained results are exemplified and commented. The paper ends with an Appendix in which the proof of Lemmas 1–2 and of Theorems 3 and 4 is given.

2. The mathematical model

Two firms X and Y produce goods which are perfect substitutes and offer them at discrete time periods $t=0,1,2,\ldots$on a common market. In order to determine the quantity of period $t+1$, the firms X and Y form expectations on the quantity of the other firm y^e_{t+1} and x^e_{t+1}, which might, for example, depend on their own quantity and the quantity of the other firm, both produced in the previous period. If we denote by x_t and y_t the output of firm X and Y at time t, respectively, the optimization problem through which the firms determine their quantities x_{t+1} and y_{t+1} are represented by $\arg\max_x \Pi_X(x_t, y^e_{t+1})$ and $\arg\max_y \Pi_Y(x^e_{t+1}, y_t)$ where $\Pi_X(\cdot,\cdot)$ and $\Pi_Y(\cdot,\cdot)$ denote the profit of firm X and Y respectively. If we assume that these optimization problems have unique solutions, then

$$x_{t+1}=r_X(y^e_{t+1}) \tag{7}$$

$$y_{t+1}=r_Y(x^e_{t+1}) \tag{8}$$

where r_X, r_Y are called Best Replies (or reaction functions). We will assume that the firms revise their beliefs according to the adaptive expectations rules

$$x^e_{t+1}=x^e_t+\alpha_1(x_t-x^e_t) \tag{9}$$

$$y^e_{t+1}=y^e_t+\alpha_2(y_t-y^e_t) \tag{10}$$

where $\alpha_i>0$ are referred to as the adjustment coefficients and we will assume the following well-known type of reaction functions:

$$r_x(y)=\mu_1 y(1-y) \tag{11}$$

$$r_y(x)=\mu_2 x(1-x) \tag{12}$$

where μ_i $(i=1,2)$ measure the intensity of the effect that one firm's actions has on the other firm. Many specifications can be found in the literature, but an analytical expression for the Best Replies is complicated. Microeconomic foundations of (11)–(12) can be found in [1].

Then, from (7) to (12) and, in order to simplify the notation, replacing x_t^e, y_t^e with x_t, y_t, the Cournot–Kopel model is obtained:

$$x_{t+1}=(1-\alpha_1)x_t+\alpha_1\mu_1 y_t(1-y_t) \tag{13}$$

$$y_{t+1}=(1-\alpha_2)y_t+\alpha_2\mu_2 x_t(1-x_t). \tag{14}$$

Firms do not change their productions according to the computed optimal productions, but they prefer to choose a weighted average between the previous production and the computed one, with weights $1-\alpha_i$ and α_i, respectively. The meaning of model implies that the economically relevant case is $\alpha_i\le 1$ $(i=1,2)$. The model (13)–(14) is a two-dimensional map, described in [1,27]. Bifurcations of map have been studied in [28,29]; fixed points, their stability and stable cycles have been studied intensively in the literature [27,30,31], in particular under the assumption

$$\mu_1=\mu_2=\mu, \quad \alpha_1=\alpha_2=\alpha, \tag{15}$$

that is the players are homogeneous with regard to the Best Replies and with regard to their expectations, respectively. These two assumptions imply that the two competitors behave identically.

In this paper, starting from the continuous time-scale counterpart of the above-mentioned non-linear Cournot duopoly game (1), we will examine the generalized model (2), where we took into account the spatial dependence, by introducing the self-diffusion and cross-diffusion terms.

3. Perturbation equations associated to the generalized model

We denote by $(\overline{u},\overline{v})$ the generic equilibrium point of (1). Besides the trivial equilibrium $(\overline{u}=0,\overline{v}=0)$, system (1) admits other non-trivial constant steady states, which can be found by solving

$$Y^3+pY+q=0 \tag{16}$$

where we have set

$$\begin{cases} Y=\mu_1\mu_2(1-\overline{u})-\dfrac{\mu_1\mu_2}{3} \\ p=-\dfrac{\mu_1^2\mu_2^2}{3}+\mu_1^2\mu_2 \\ q=-\dfrac{2\mu_1^3\mu_2^3}{27}+\dfrac{\mu_1^3\mu_2^2}{3}-\mu_1^2\mu_2. \end{cases} \tag{17}$$

The solutions to (16) are

$$\begin{cases} Y_1=Y_++Y_- \\ Y_2=-\dfrac{Y_++Y_-}{2}+i\dfrac{Y_+-Y_-}{2}\sqrt{3} \\ Y_3=-\dfrac{Y_++Y_-}{2}-i\dfrac{Y_+-Y_-}{2}\sqrt{3} \end{cases} \tag{18}$$

where

$$Y_+=\sqrt[3]{-\frac{q}{2}+\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}, \quad Y_-=\sqrt[3]{-\frac{q}{2}-\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}, \tag{19}$$

hence $(\overline{u},\overline{v})=\left(\frac{2}{3}-Y_i/\mu_1\mu_2,\ \frac{2}{9}+Y_i/\mu_1-Y_i^2/\mu_1^2\mu_2\right)$, with $(i=1,2,3)$.

Precisely, if $\overline{\Delta}=q^2/4+p^3/27$ is non-positive (positive), which implies $-\mu_1^2\mu_2^2+4\mu_1\mu_2^2-18\mu_1\mu_2+4\mu_1^2\mu_2+27$ non-positive (positive), then three (only one) real solutions are created. Let us set

$$U=u-\overline{u}, \quad V=v-\overline{v}, \tag{20}$$

then, the perturbation equations associated to (2) are given by

$$\begin{cases} \partial_t U=-\alpha_1 U+\alpha_1\mu_1(1-2\overline{v})V+\gamma_{11}\Delta U+\gamma_{12}\Delta V-\alpha_1\mu_1 V^2 \\ \partial_t V=\alpha_2\mu_2(1-2\overline{u})U-\alpha_2 V+\gamma_{21}\Delta U+\gamma_{22}\Delta V-\alpha_2\mu_2 U^2. \end{cases} \tag{21}$$

Denoting by

$$\begin{cases} f(U,V)=-\alpha_1\mu_1 V^2, & g(U,V)=-\alpha_2\mu_2 U^2 \\ a_{11}=-\alpha_1, & a_{22}=-\alpha_2 \\ a_{21}=\alpha_2\mu_2(1-2\overline{u}), & a_{12}=\alpha_1\mu_1(1-2\overline{v}) \end{cases} \tag{22}$$