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# Non-linear stability analysis of laminated composite plates on one-sided foundation by hierarchical Rayleigh–Ritz and finite elements

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ABSTRACT

In this paper, the influence of one-sided foundation on the unilateral buckling behavior of laminated composite orthotropic plates is investigated under compressive load. Derivation of governing equations is based on Kirchhoff's hypotheses and the principle of minimum total potential energy. The solutions are performed by the hierarchical Rayleigh–Ritz (HRRM) and finite element methods (HFEM) and are compared. Most of previous research studies on the unilateral buckling of plates are limited to single-layer plates. The results show that unsymmetric lamination experiences lower critical loads than those of symmetric lamination due to the existence of extensional–bending coupling in unsymmetric laminated sequencing schemes. Influences of aspect ratio, fiber orientation, the number of plies, Young's modulus ratio, and different boundary conditions on the unilateral buckling load are examined. The numerical results are validated with previous works studying unilateral buckling of single-layer plates resting on one-sided foundation.



## 1. Introduction

Plates, as one of the most commonly used structural elements of mechanical systems and load-bearing structures, are among the most important structural parts. They have the widest utility in structural, mechanical, and aerospace engineering, for building flexible structures, such as aircraft, spacecraft, and marine structures. From a mechanical point of view, whenever a plate is subjected to compressive in-plane loads and tends to go out of the loaded plane, the buckling phenomenon has occurred. The buckling, which occurs as a result of compressive in-plane loads, structural defects, eccentric loads, etc., leads to structural instability and damage to structures.

The unilateral buckling of plates is in fact a contact problem. Plates resting on foundations, e.g., plates reinforced with concrete beams/columns are some of the practical usages of stability analysis studied in this research. The existence of a unilateral constraint is enough justification for a plate to show a nonlinear behavior. Many studies were conducted on the development of computational and theoretical methods for the analysis of elastic bodies in contact. In these studies, it was assumed that one of the bodies would act as the foundation of the other one, and cohesive forces would keep together the bodies. Therefore, since the contact between the two bodies was known to be bilateral, formulating the model of problem by linear differential equations was possible. In many cases, because of the inability of a semi-rigid surface for producing cohesion, it cannot act as a tension generating constraint. This kind of boundary condition is known as a unilateral contact condition. The dual nature of the foundation and unknown contact area between two bodies in contact would make the problem even more complicated. Regarding these conditions, the analysis of unilateral systems is of more numerical nature. Therefore, the study on the unilateral buckling of plates is restricted to the last forty years.

In 1994, Shawan and Wass [1] obtained the critical buckling load of a rectangular elastic plate with a finite length by Galerkin's method based on the classic plate theory and choosing an appropriate elastic energy function for the elastic foundation. In 1997, Smith et al. [2] studied the stability of constrained rectangular plates under shear loading by using the Rayleigh–Ritz method. They modeled the unilateral constraint as a one-sided foundation and studied the behavior of plate with different boundary conditions for a shear loading case. In 1998, Shahwan and Wass [3] investigated the unilateral buckling of constrained plates using Galerkin's method and considering Kirchhoff's plate theory. They used a non-linear elastic foundation, which was called a one-sided foundation. In 1999, Smith et al. [4] analyzed

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**Nomenclature**

| | |
|---|---|
| $a$, $b$, $h$ | length, width and thickness of plate respectively |
| $a/b$ | aspect ratio |
| $N$ | number of lamina |
| $z_k$ | thickness of the $k$th lamina |
| $u, v, w$ | displacement field components in $x$, $y$ and $z$ global coordinate system respectively |
| $u_0, v_0, w_0$ | mid-plane displacement field components in the $x$, $y$ and $z$ directions respectively |
| $\{\overline{\varepsilon}\}$ | total strain vector |
| $\{\overline{\varepsilon}^{(01)}\}$ | linear part of membrane strain vector |
| $\{\overline{\varepsilon}^{(02)}\}$ | non-linear part, the von Karman's strain vector |
| $\{\overline{\varepsilon}^{(1)}\}$ | bending strain vector |
| $Q_{ij}, \overline{Q}_{ij}$ | the 2D-stiffness and transformed reduced stiffness of the $k$th lamina |
| $E_1, E_2$ | Young's moduli along the fiber direction and normal to the fiber in 1–2 material coordinate system |
| $G_{12}$ | shear modulus in the 1–2 plane |
| $\upsilon_{12}, \upsilon_{21}$ | Poisson's ratio**s** in the 1–2 plane |
| $\sigma_i, \varepsilon_i \quad i=1,2,6$ | laminate stress and strain vector components |
| $\{N\}$ | total force resultant vector of in-plane stresses |
| $\{M\}$ | total moment resultant vector of out-of-plane stresses |
| $[A]$, $[B], [D]$ | the extensional, bending–extensional coupling, and bending stiffnesses matrices |
| $\varphi_m(\xi,\eta)$ | $m$th term of polynomial for out-of-plane displacement field |
| $n$ | number of two-dimensional polynomial terms for out-of-plane displacement field |
| $\phi_b(\xi,\eta)$ | out-of-plane boundary conditions polynomial for hierarchical Rayleigh–Ritz method |
| F, S, C | free, simple and clamped edge boundary conditions |
| $\xi, \eta$ | dimensionless coordinates (natural coordinates) |
| $x$, $y$ | Cartesian coordinates |
| $\theta$ | lamina fiber orientation angle with respect to $x$ axis |
| $\{U\}^R$ | plate displacement generalized coordinate vector for hierarchical Rayleigh–Ritz method |
| $\{U\}^e$ | element displacement field vector for the hierarchical finite element method |
| $\{\hat{U}\}^R$ | plate total unknown coefficient vector (generalized coordinate) for hierarchical Rayleigh–Ritz method |
| $\{\hat{U}\}^e$ | element unknown coefficient vector for hierarchical finite element method |
| $\{\hat{U}\}$ | unknown coefficient vector |
| $[N]^R$ | plate displacement field matrix for hierarchical Rayleigh–Ritz method |
| $[N]^{FE}$ | element displacement field matrix for hierarchical finite element method |
| $[C_M]^R$ | matrix resulting from membrane strain vector for hierarchical Rayleigh–Ritz method |
| $[C_M]^{FE}$ | matrix resulting from membrane strain vector for hierarchical finite element method |
| $[C_M]$ | matrix resulting from membrane strain vector |
| $[C_B]^R$ | matrix resulting from bending strain vector for hierarchical Rayleigh–Ritz method |
| $[C_B]^{FE}$ | matrix resulting from bending strain vector for hierarchical finite element method |
| $[C_B]$ | matrix resulting from bending strain vector |
| $H_i(\xi)\ ,i=3,\dots,\ 10$ | hierarchical functions |
| $H_i(\xi)\ ,i=1,2$ | first-order Lagrange functions |
| $g_{ij}$ | Hermite interpolation functions |
| $\delta\, U_e$ | potential energy variations resulting from deformation of springs |
| $\delta V$ | potential energy variations in plate resulting from in-plane loads |
| $\delta U$ | strain energy variations in laminated composite plate |
| $\delta \Pi$ | total Potential energy variations |
| $[K_e]$ | stiffness matrix resulting from elastic foundation |
| $X$ | contact function |
| $k_f$ | spring stiffness |
| $[K_T]$ | total stiffness matrix |
| $[K_{mm}]$ | stiffness matrix resulting from membrane loads |
| $[K_{mb}]$ | stiffness matrix resulting from coupling loads |
| $[K_{bb}]$ | stiffness matrix resulting from bending loads |
| $[K_G]$ | geometrical stiffness matrix |
| $[H]$ | matrix resulting from von Karman's strain |
| $[H]^{FE}$ | matrix resulting from von Karman's strain for the hierarchical finite element method |
| $[H]^R$ | matrix resulting from von Karman's strain for the hierarchical Rayleigh–Ritz method |
| $[N_0]$ | matrix resulting from in-plane forces |
| $\lambda$ | critical bucking load |
| $B.C.$ | boundary condition |

the stability of rectangular constrained plates by the Rayleigh–Ritz method. They verified the simulated results by some experiments in the same year [5]. In 2000, Bradford et al. [6] studied the buckling of thin-walled semi-compact steel plates resting on a rigid medium subjected to bending, compression and shear. Plates, which experience the first yield before local buckling, are referred to as semi-compact. In this paper based on the energy method, the limiting depth to thickness ratios were obtained that determine the semi-compact section classification. In 2007, Hedayati et al. [7] investigated the stability of plates, reinforced with concrete beams, which was one of the practical applications of a unilateral buckling problem. They analyzed the local stability of a supporting plate, reinforced and bolted to a concrete foundation, by using Lagrange's multipliers for introducing the constraints and solved it by the Rayleigh–Ritz method. They modeled the concrete support as a one-sided foundation. In 2009, Muradova et al. [8] conducted a number of studies on buckling of rectangular plates constrained bi-laterally by a foundation. The foundation was modeled with two different mechanisms of Winkler and one-sided nature. In 2011, Ma et al. [9] accomplished a research on the buckling of an infinite thin plate resting on a one-sided foundation subjected to shear forces. The infinite plate was modeled as a 1D structural system. In that study, they calculated the contact area using the concept of lateral buckling mode functions and achieved two non-linear governing equations for two different regions of in-contact and non-contact areas.

Considering some challenges such as fuel economy, and reaching higher speeds, an increasing interest in reducing the thickness and weight of structural parts in aerospace, marine and automotive industries has led the practitioners to replace the traditional isotropic materials with advanced composite materials, which have offered both higher specific strength and lower specific weight advantages. Among different issues of advanced composite structures, the buckling behavior of the composite laminated plates has attracted the attention of many researchers. In general, an analytical or exact solution of this kind of plate problem, because of non-linear behavior, is restricted to some limited types of boundary conditions, loadings, and layer sequence schemes. A large body of studies in the field of analytical solutions of laminated composite plates has been conducted by Reddy [10]. For