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On the contact problem of an inflated spherical hyperelastic membrane

Nirmal Kumar*, Anirvan DasGupta

Department of Mechanical Engineering and Centre for Theoretical Studies, Indian Institute of Technology Kharagpur, Kharagpur 721302, India

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ABSTRACT

In this paper, the mechanics of contact of an inflated spherical non-linear hyperelastic membrane pressed between two rigid plates has been studied. We have considered the membrane material to be a homogeneous and isotropic Mooney–Rivlin hyperelastic solid. All three cases, namely frictionless, no-slip and stick–slip conditions have been considered separately in the plate-membrane contact region. The stretch of the membrane, and the surface traction (for no-slip contact) has been determined. For the stick–slip case, the sliding front is observed to be initiated at the contact periphery which moves towards the pole. The state at which the impending wrinkling condition occurs has been determined analytically. It is observed that the impending wrinkling state occurs at the periphery of the contact. Based on this, the minimum initial stretch (inflation) required to prevent wrinkling at any point in the membrane has been determined.

1. Introduction

Closed inflated membranes are routinely used in airbags and suspensions for cushioning and absorbing shocks. Such applications fundamentally involve a variable contact between an inflated membrane and a rigid or flexible surface. The membrane has the characteristics of a non-linear spring. The contact problem is interesting and challenging due to large deformations which involve geometric non-linearity, material non-linearity and complex contact conditions due to friction and adhesion. This work is motivated by some of these fundamental issues in the mechanics of deformation of inflated membranes with contact. Some large deformation problems without contact can be found in [1–7]. Models for contact problem with or without friction and adhesion can be found in [8–22]. Modeling of membranes with wrinkling has been discussed in detail in [23–28].

An approach based on the conversion of a boundary value problem into an initial value problem for axisymmetric membrane problems has been developed by Yang and Feng [29]. This method has been extended by Patil and DasGupta [30] for the free inflation problem. Feng and Yang [31] investigated the deformation of pressurized spherical balloon kept between two large rigid plates considering Mooney–Rivlin model. However, they have not considered adhesion and friction between the membrane and the plate during contact. Another frictionless contact problem has been addressed by Nadler [32] in which the condition and domain of wrinkling has been analyzed assuming that the wrinkling can occur only in contacting region. Charrier and Shrivastava [33,34] have related thermoforming with inflation of axisymmetric and non-axisymmetric membranes against a rigid contact (cylindrical, conical and flat) with no-slip, and no-friction conditions. Recently, Long et al. [35] have considered the adhesive contact problem of circular membranes with a flat rigid plate. However, for no-slip condition they have used an approximated value of the infinitesimal increment in contact radius without considering the thermodynamic properties of the gas inside the membrane. In our problem, we have considered closed system instead of open system considered in [35]. In no-slip case, we found increment in contact radius by considering thermodynamic properties of the gas inside the membrane. The contact problem of an uninflated membrane with frictional sliding condition has been solved using the finite element method in [36,37].

In the existing literature, while free inflation problems have been well studied, the contact problem of inflated membranes has not been explored completely. In particular, contact problems with stick–slip contact has not been solved yet. Contact problems of an inflated membrane considered as a closed thermodynamic system remains to be addressed. Further, the occurrence of impending wrinkling condition in such problems is also of importance which has not been discussed in detail.

In the present work, we are interested in understanding the mechanics of deformation of inflated membranes with contact. To keep the geometry simple, we have considered a completely spherical inflated membrane pressed symmetrically between two rigid plates. We have formulated the problem as a variational

* Corresponding author. Tel.: +91 933 384 6992; fax: +91 322 225 5303.
E-mail addresses: nirmal.mit@gmail.com, nirmaldilber@yahoo.co.in (N. Kumar), anir@mech.iitkgp.ernet.in (A. DasGupta).

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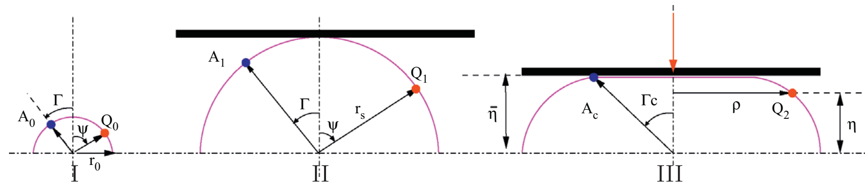

Fig. 1. Geometry of spherical membrane before and after inflation and contact with two large rigid plates.

problem using the Mooney–Rivlin strain energy function for the membrane and the potential energy of the inflating gas. The deformation process is assumed to be isothermal. We have studied the problem with frictionless, no-slip and stick–slip contact conditions. The stick–slip case is actually a non-conservative problem, and does not have a variational formulation as such. However, as proposed in this paper, one can formulate a sequence of variational problems while checking and allowing for intermediate slip using the force equilibrium condition of the membrane at the contact. The stretches and contact traction (for no-slip contact) have been determined. The occurrence of the impending wrinkling state has also been determined analytically to find the limits of the solutions obtained with the present formulation.

2. Problem formulation

Consider a spherical balloon of uninflated radius r_0 and uniform thickness h_0 (state I), which is inflated to a radius r_s (state II) by a pressure p_0. Only the upper half of the balloon is shown in Fig. 1. Consider two rigid plates coming in contact and pressing the inflated balloon symmetrically, as shown in Fig. 1 (state III). The material points of the membrane are parameterized through the angles θ (azimuthal angle) and ψ. The extent of contact is measured by the contact angle Γ_c (see point A_c in Fig. 1 state III) in deformed membrane, which corresponds to the material points with $\psi=\Gamma$ (see points A_0 and A_1 in Fig. 1 states I and II, respectively). We assume that the problem is axisymmetric and the membrane material is homogeneous and isotropic. Further, all thermodynamic processes are assumed to be reversible and isothermal during inflation and contact.

2.1. Kinematics of deformation

The principal stretch ratios for the membrane is written as

$$\lambda_1=\frac{\sqrt{\rho'^2+\eta'^2}}{r_0},\quad \lambda_2=\frac{\rho}{r_0\ \sin\ \psi},\quad \lambda_3=\frac{r_0^2\ \sin\ \psi}{\rho\sqrt{\rho'^2+\eta'^2}} \tag{1}$$

where we have used the incompressibility condition $\lambda_1\lambda_2\lambda_3=1$. The prime denotes derivative with respect to ψ. The strain invariants in terms of the stretch ratios are given by

$$I_1=\lambda_1^2+\lambda_2^2+\lambda_3^2 \quad \text{and} \quad I_2=\frac{1}{\lambda_1^2}+\frac{1}{\lambda_2^2}+\frac{1}{\lambda_3^2} \tag{2}$$

2.2. Potential energy functional

For a Mooney–Rivlin material, the strain energy density function may be written as

$$\hat{U}=C_1(I_1-3)+C_2(I_2-3)=C_1[(I_1-3)+\alpha(I_2-3)], \tag{3}$$

where C_1 and C_2 are material constants and $\alpha=C_2/C_1$. The strain energy of the membrane is then obtained as

$$U=\int_V \hat{U}\,dV=2r_0^2h_0\int_0^{2\pi}\int_0^{\pi/2}\hat{U}\ \sin\ \psi\ d\psi\ d\theta. \tag{4}$$

The internal energy of the inflating gas can be written as

$$W_p=-2\int_0^{2\pi}\int_0^{\pi/2}\frac{p}{2}(\eta'\rho^2)\ d\psi\ d\theta. \tag{5}$$

The total potential energy function is given by $\Pi=U-W_p$.

2.3. Governing equation

In the following, we derive the governing equations for the non-contacting region $\psi\epsilon[\Gamma,\pi/2]$, and the contacting region $\psi\epsilon[0,\Gamma]$ (discussed later) separately. Using a substitution $\lambda_1=\sqrt{(\lambda'_2\ \sin\ \psi+\lambda_2\ \cos\ \psi^2+\eta'^2}$ one can express the potential energy as a function of λ_2, $\lambda_2{}'$ and η'. The governing equations for the non-contacting region can be obtained as

$$\frac{d}{d\psi}\frac{\partial\Pi}{\partial\lambda'_2}-\frac{\partial\Pi}{\partial\lambda_2}=0,\qquad \frac{d}{d\psi}\frac{\partial\Pi}{\partial\eta'}=0 \tag{6}$$

Defining $w=-\eta'$, and $v=\lambda'_2 r_0\ \sin\ \psi$ in Eq. (6), the governing equations for the non-contacting region read

$$v'=f_1(\lambda_2,v,w,\psi), \tag{7}$$

$$w'=f_2(\lambda_2,v,w,p,\psi), \tag{8}$$

$$\lambda'_2=\frac{v}{r_0\ \sin\ \psi}. \tag{9}$$

The boundary conditions for the non-contacting region are

$$v|_{\psi=\Gamma}=v_0,\quad v|_{\psi=\pi/2}=0,\quad w|_{\psi=\Gamma}=0,\quad \lambda_2|_{\psi=\Gamma}=\lambda_c, \tag{10}$$

where v_0 and λ_c are as yet unknown. They will be decided from the junction condition discussed later.

The pressure and volume inside the contact free spherical membrane are related to λ_s as

$$V_0=\frac{4}{3}\pi r_0^3\lambda_s^3,\quad p_0=4C_1\left(1-\frac{1}{\lambda_s^6}\right)(1+\alpha\lambda_s^2)\frac{h_0}{r_0\lambda_s}, \tag{11}$$

where $\lambda_s=r_s/r_0$ and V_0 is volume of the spherical membrane after inflation and before contact. The volume inside the membrane after contact is given by (using Eq. (1))

$$V=2\pi\int_0^{\overline{\eta}}\rho^2 d\eta=2\pi\int_\Gamma^{\pi/2}(r_0\lambda_2\ \sin\ \psi)^2\eta' d\psi \tag{12}$$

where $\overline{\eta}$ is vertical distance of the rigid plate from the equator of the balloon after contact. Assuming isothermal compression of the gas. The pressure after contact may be written as $p=p_0V_0/V$.

2.4. Contact conditions

We have considered three types of contact conditions between the plate and the membrane, namely frictionless contact, no-slip contact and stick–slip contact. In the frictionless contact, the material points of the membrane in the contacting region can slide freely over the plate. The governing equations are considered separately in the non-contacting and contacting regions. In the contacting region, the pressure work W_p vanishes because $w=0$ throughout. Hence, the governing equation in this region is given