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Dispersive terms in the averaged Lagrangian method

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A B S T R A C T

In this paper we develop Whitham's formalism of the averaged Lagrangian in the problem of Stokes waves on the surface of a layer of ideal fluid by taking into account dispersive terms. We derive a general expression for the Lagrangian in which Whitham's term with the non-linear frequency of narrow-band wave trains is expressed in explicit form using derivatives of the complex amplitude phase of the first harmonic envelope. This Lagrangian form simplifies derivation of the evolution equations as variational equations.

1. Introduction

Whitham's averaged Lagrangian method [1–5] is used in non-linear wave theory to obtain evolutionary equations, together with the multiple scales method and Zakharov's Hamiltonian formalism. Whitham formulated a procedure to describe non-linear waves using a Lagrangian expanded in terms of non-linearity and averaged over fast oscillations. He obtained the averaged Lagrangian for waves on the surface of a ideal liquid [2]. The focus of that work was on weak non-linearity and not dispersion. Thus, the Lagrangian obtained contains only an expansion in powers a^2 and a^4 of the amplitude a of the complex-valued amplitude of envelope $\mathcal{A} = a\exp(i\vartheta)$ of the fundamental harmonic of the free surface elevation, and does not include derivatives of a for the coordinates [2]. The derivatives of ϑ in the Lagrangian can be obtained, albeit implicitly, by means of the functions of coordinates $\omega(x,t)$, $k(x,t)$.

Owing to the absence of derivatives of a, variational equation with respect to a give the first correction to the dispersion relation for non-linearity (identical to coefficient by the non-linear term of the non-linear Schrödinger equation, NSE). However, this lacks the necessary dispersive part and an adequate combination of variation equations with respect to a and ϑ [compare Eq. (41), (42)]. Thus, the first term a_{xx} is not sufficient in the expression $(a_{xx} - a\vartheta_x^2 + i(2a_x\vartheta_x + a\vartheta_{xx}))\exp(i\vartheta)$ for coincidence with $\mathcal{A}_{xx}$ and the identity of this combination to NSE. As a result, coefficients for NSEs and extensions to various waves using the averaged Lagrangian have been derived by omitting the dispersive term of the type A_{xx}. This approach is based on the assumption that the coefficient for the NSE non-linear term coincides with that for the non-linear term of the non-linear dispersion relation and can be obtained using a single variational equation with respect to the amplitude a.

The method for constructing the averaged Lagrangian for identity between the combination of the variational equations in Whitham's theory and the NSE was extended for waves on a fluid surface in the specific case of a basin with infinite depth [6,7]. In recent work, this extension was generalized to the case of a basin with arbitrary depth [8]. In this case the formulas are much more awkward because of the dependence of all values on the layer thickness. Moreover, the averaged Lagrangian obtained depends on new functions—the amplitudes of the zero harmonics of the velocity potential and surface elevation—in addition to the complex-valued amplitude of the fundamental harmonic $\mathcal{A}$. The corresponding variational equations lead to a system of three evolutionary equations for these variables, the Benney–Roskes equations [9] instead of the NSE for $\mathcal{A}$. These extensions have shown that the dispersive terms of the averaged Lagrangian must be taken into account. The solution proposed for this problem consists of (i) addition of supplementary trial functions and (ii) consideration of the slow dependence of the amplitudes of harmonics on x and t.

However, the only explicit forms for the complex-valued amplitude of envelope $\mathcal{A}$ in the Lagrangian are derivatives of a, and derivatives of the phase ϑ are only included implicitly through the functions $\omega(x,t)$ and $k(x,t)$ [6–8]. Therefore, a and ϑ are not treated equally. The aim of the present study was to introduce

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derivatives of the phase ϑ into the Lagrangian instead of $\omega(x,t)$ and $k(x,t)$ without loss of precision. Besides allowing parity between a and ϑ, this simplifies derivation of the evolution equations from this Lagrangian.

2. Formulation of the problem for waves on a fluid surface

2.1. Luke's Lagrangian

According to the Hamilton principle, the Lagrangian for waves on a fluid surface is equal to the difference between the kinetic and potential energies of the fluid. The boundary conditions are introduced by the constraints [10]

$$L=\int_{-h}^{\eta(x,t)}\left(\frac{1}{2}\left(\varphi_x^2+\varphi_y^2\right)-gy\right)dy+\text{constraints}, \tag{1}$$

where $\eta(x,t)$ is a profile of the fluid surface (a wave), φ is the fluid velocity potential, x and y are the horizontal and vertical coordinates, respectively, h is the fluid depth, and g is gravitational acceleration. Luke demonstrated that for a vortex-free fluid, the following more convenient form of the Lagrange function can be used instead of (1) [10]:

$$L=\int_{-h}^{\eta(x,t)}\left(\varphi_t+\frac{1}{2}\left(\varphi_x^2+\varphi_y^2\right)+gy\right)dy, \tag{2}$$

provided that averaging is carried out over the period of fast oscillations.

2.2. Expansion of the trial functions for Stokes waves and Lagrangian calculation

In the linear approximation, the equations of motion for a wave on the surface of an ideal fluid allow harmonic waves $\eta(x,t)=\varepsilon a\cos\theta_0, \theta_0=k_0x-\omega_0 t$ for the free surface elevation and $\varphi(x,y,t)=\varepsilon(\psi+Af_1(y)\sin\theta_0), f_1(y)=\cosh(k_0(y+h))$ for the velocity potential, where a, A, ψ, wavenumber k_0 and frequency ω_0 do not depend on the coordinates. Taking into account the first nonlinear terms in the equations of motion leads to the appearance of zero and second harmonics, and to slow dependence of all amplitudes and k and ω on the coordinates

$$\eta_{Wh}=\varepsilon a\cos\theta+\varepsilon^2(b+a_2\cos 2\theta), \tag{3}$$

$$\varphi_{Wh}=\varepsilon(\psi+Af_1(y)\sin\theta)+\varepsilon^2A_2f_2(y)\sin 2\theta, \tag{4}$$

$$f_2(y)=\cosh(2k_0(y+h)),$$
$$\theta=k(x,t)x-\omega(x,t)t.$$

Here b is the zero harmonic of the wave profile and ψ is the zero harmonic of the velocity potential, which both slowly vary in time and space x, similar to the amplitudes of other harmonics, and describe the modulations of fast oscillations. Eqs. (3) and (4) were used by Whitham as trial functions [2,5], but the slow dependence of amplitudes A and A_2 on coordinates is not taken into consideration [2, Eq. (14)]).

We introduce the rapid oscillations $\exp i(k_0x-\omega_0t)$ in (3) explicitly with the aim of applying the averaged Lagrangian to construct variational equations describing the slow evolution the amplitude of the oscillations:

$$a(x,t)\cos(k(x,t)x-\omega(x,t)t)=\tfrac{1}{2}\mathcal{A}(x,t)e^{i(k_0x-\omega_0t)}+c.c.,$$

where $\mathcal{A}(x,t)=a(x,t)\exp i\vartheta(x,t)$ is the complex-valued amplitude with modulus $a(x,t)$ and phase

$$\vartheta(x,t)=(k(x,t)-k_0)x-(\omega(x,t)-\omega_0)t \tag{5}$$

of the envelope of the fundamental harmonic of the rapidly oscillating carrier wave η with internal filling $\exp i(k_0x-\omega_0t)$.

The first improvement to Whitham's theory takes into account the corrections δ_1 and δ_2 to the fundamental harmonics in the trial functions. These corrections have the same order of magnitude ε^2 that is also the second harmonic [11]:

$$\eta=\eta_{Wh}+\delta_1,\quad \delta_1=\varepsilon^2(\tilde{a}\sin\theta+\tilde{\tilde{a}}\cos\theta), \tag{6}$$

$$\varphi=\varphi_{Wh}+\delta_2,\quad \delta_2=\varepsilon^2f_3(y)(\tilde{A}\cos\theta+\tilde{\tilde{A}}\sin\theta),$$

$$f_3(y)=h\tanh k_0h\cosh(k_0(y+h))-(y+h)\sinh(k_0(y+h)). \tag{7}$$

The vertical dependence of the correction δ_2 for the velocity potential on $f_3(y)$ is noted from the solution of Laplace's equation for φ, taking into account the introduction of slow coordinates and the representation of φ in the form of a series [12,13]. Moreover, the coefficient $f_3(y)$ in δ_2 is chosen in line with the limiting case of infinite depth [12].

For completeness of the trial functions, the terms $\tilde{\tilde{a}}\cos\theta$ in (6) and $\tilde{\tilde{A}}\sin\theta$ in (7) are included. It is evident in Eqs. (25) and (27) below that the amplitudes $\tilde{\tilde{a}}$ and $\tilde{\tilde{A}}$ are connected by derivatives of the phase ϑ of the complex-valued envelope $\mathcal{A}(x,t)=a(x,t)\exp i\vartheta(x,t)$ of the wave η, namely, $\tilde{\tilde{a}}=-a\vartheta_t/\omega_0$ and $\tilde{\tilde{A}}=-A\vartheta_x$. These do not appear explicitly in the original extensions of Whitham's work [6–8], since they contain derivatives of ϑ in the averaged Lagrangian. Specifically, to express the averaged Lagrangian in terms of derivatives of ϑ explicitly for consistency with a, the trial functions $\tilde{\tilde{a}}$ and $\tilde{\tilde{A}}$ in (6) and (7) are supplemented. The amplitudes $\tilde{A}$, $\tilde{\tilde{A}}$, $\tilde{a}$, and $\tilde{\tilde{a}}$ of the corrections δ_1 and δ_2 should be determined according to the variation of the Lagrangian with respect to them.

The second revision of Whitham's theory considers the slow dependence of A, A_2, $\tilde{A}$ and $\tilde{\tilde{A}}$ on x,t in (2). We take them into account in the derivatives φ_t and φ_x of (7) as corrections of magnitude Δ_1 and Δ_2 to the derivatives $-\omega\varphi_\theta+\varepsilon^2\psi_t$ and $k\varphi_\theta+\varepsilon^2\psi_x$ [2,5]:

$$\varphi_t=-\omega\varphi_\theta+\varepsilon^2\psi_t$$
$$+\Delta_1,\quad \Delta_1=\varepsilon(\varphi_AA_t+\varphi_{A_2}A_{2t}+\varphi_{\tilde{A}}\tilde{A}_t+\varphi_{\tilde{\tilde{A}}}\tilde{\tilde{A}}_t), \tag{8}$$

$$\varphi_x=k\varphi_\theta+\varepsilon^2\psi_x+\Delta_2,\quad \Delta_2=\varepsilon(\varphi_AA_t+\varphi_{A_2}A_{2x}+\varphi_{\tilde{A}}\tilde{A}_t+\varphi_{\tilde{\tilde{A}}}\tilde{\tilde{A}}_t). \tag{9}$$

This extension Whitham's method previously was used to obtain evolutionary equations by the variational procedure for waves in a collisionless plasma [14] and waves on the surface layer of a liquid [15]. In the deep-water limit (in which ψ and b vanish), both extensions were used by Yuen and Lake [6,7].

Substitution of (8) and (9) and φ_y into (2) gives the following expression for the Lagrangian:

$$L=-(\omega-\varepsilon^2k_0\psi_x)\int_{-h}^{\eta}\varphi_\theta\,dy+\frac{1}{2}\int_{-h}^{\eta}\left(k^2\varphi_\theta^2+\varphi_y^2\right)dy$$
$$+\int_{-h}^{\eta}\left(\Delta_1+k_0\varphi_\theta\Delta_2+\varepsilon^2\psi_x\Delta_2+\frac{1}{2}\Delta_2^2\right)dy+m_3+m_4, \tag{10}$$

$$m_3=\varepsilon^2(\psi_t+\tfrac{1}{2}\varepsilon^2\psi_x^2)(h+\eta),\quad m_4=\tfrac{1}{2}g\left(\eta^2-h^2\right).$$

Eq. (10) has a new term (third integral) that takes into account the dependence of A, A_2, $\tilde{A}$ and $\tilde{\tilde{A}}$ on the coordinates, as well as the corrections δ_1 (6) in η and δ_2 (7) in φ. In the term $k_0\varphi_\theta\Delta_2$ (as $\Delta_2\sim\varepsilon$) and hereafter, the wavenumber k is replaced by the wavenumber k_0 for high-order terms in ε.

The trial functions η and φ of the Lagrangian contain supplementary amplitudes $\tilde{\tilde{a}}$ and $\tilde{\tilde{A}}$ according to (6) and (7) and this leads to explicit occurrence of ϑ_x and ϑ_t in (10) by replacing ω and k according to (11).