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Inertia effect on the onset of convection in rotating porous layers via the "auxiliary system method"☆

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ABSTRACT

Via the *auxiliary system method* (Rionero, 2012 [35] and Rionero, 2013 [36,37]) the onset of convection in rotating porous layers in the presence of inertia is investigated. The effects of rotation and inertia are respectively measured through the Taylor number $\mathcal{T}$ and Vadasz number V_a (Section 2). For the tridimensional perturbations and the full non-linear problem, it is shown that:

(a) there exists a critical Taylor number $\mathcal{T}_c \approx 1.53$ such that for $\mathcal{T} \le \mathcal{T}_c$ the inertia has no effect on the onset of convection;
(b) for $\mathcal{T} > \mathcal{T}_c$ there exists an associate critical Vadasz number $V_a^{(c)}(\mathcal{T})(>0)$ such that, only for $V_a < V_a^{(c)}(\mathcal{T})$, the inertia has effect on the onset of convection, and only in this case the convection arises via an oscillatory motion (cf. Theorems 5.2 and 5.3);
(c) subcritical instabilities do not exist;
(d) the global non-linear stability is guaranteed by the linear stability;
(e) also in the case $\{\mathcal{T} > \mathcal{T}_c, V_a < V_a^{(c)}\}$ the critical Rayleigh number can be given in closed form.

1. Introduction

The fluid motions in rotating porous layers, because of the numerous applications in the real world (food process industry, chemical porous industry, crystal growth, thermal engineering, rotating machinery) – in the past as nowadays – has attracted the attention of numerous researchers, either in the absence or in the presence of inertia (cf. [1–37] and the references therein). The paper [16] of Vadasz, which motivates the present paper, contains particularly interesting results. In that paper the full non-linear equations for convection in a saturated rotating porous medium, in the presence of inertia, are derived and the linear instability together with the *weak non-linear stability* (with respect to bi-dimensional perturbations) of the thermal conduction solution, are investigated.[1] Precisely: (1) the linear instability is studied via the normal modes (Chandrasekhar method [2]) and the critical Rayleigh number for stationary convection is obtained in closed form; (2) conditions for overstable convection are obtained; (3) a weak non-linear analysis is performed.

Our aim in the present paper is to reconsider the problem investigated in [16], according to the *auxiliary system method* introduced by Rionero in [27–37], formalized in [35] for ternary porous mixtures and generalized in [36] to porous mixtures with any number of salts (cf. Appendix B). In fact we provide a direct application of the Rionero approach for the general tridimensional perturbations, in the case of the full non-linearities. Denoting by $\mathcal{T}$ the Taylor number (Section 2) and by R_c and $R_c^{(r)}$ the critical Rayleigh number in the presence and in the absence of inertia, respectively, for a rotating porous layer heated from below, we show that:

(i) there exists a critical Taylor number $\mathcal{T}_c \approx 1.53$ such that

$$\mathcal{T} \le \mathcal{T}_c \Rightarrow R_c = R_c^{(r)}, \quad \forall V_a \in \mathbb{R}^+; \tag{1.1}$$

(ii) there exists a critical Vadasz number $V_a^{(c)}$ such that

$$\{\mathcal{T} > \mathcal{T}_c, V_a > V_a^{(c)}\} \Rightarrow R_c = R_c^{(r)}; \tag{1.2}$$

(iii) the inertia term has influence on the onset of convection only for $\{\mathcal{T} > \mathcal{T}_c, V_a < V_a^{(c)}\}$. In fact not only $R_c < R_c^{(r)}$ for such values

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[1] In the Vadasz equations the inertia term is multiplied by a non-dimensional number $1/V_a$ with V_a named Vadasz number in [24] (cf. Appendix A).

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of $\mathcal{T}$ and V_a, but unlike the cases (i)–(ii) (which are cases of stationary convection), the convection arises via an oscillatory motion (overstable convection);
(iv) subcritical instabilities do not exist;
(v) the global non-linear stability is guaranteed by the linear stability;
(vi) also in the case $\{\mathcal{T} > \mathcal{T}_c, V_a < V_a^{(c)}\}$ the critical Rayleigh number is given in closed form.

We remark that, as far as we know, for tridimensional perturbations and the strong non-linearities, the properties (i)–(vi) are new in the existing literature and furnish new contributions which ratify, enrich and complete the behaviors obtained in [16].

The plan of the paper is the following. Section 2 is devoted to some preliminaries (in particular the Vadasz model is recalled). In Section 3 it is shown that the independent fields are only three. Section 4 is devoted to linear stability while critical Rayleigh number is studied in Section 5. The absence of subcritical instabilities and the global non-linear stability are analyzed in the subsequent sections (Sections 6 and 7). Precisely, Section 6 is devoted to the non-linear equation governing each Fourier component of the perturbations, while in Section 7 the absence of subcritical instabilities and global non-linear stability are investigated. Some final remarks are concentrated in Section 8. The paper ends with an Appendix A in which are recalled: (A) the original Vadasz equation; (B) the essential guidelines of the Rionero method followed in the present paper; (C) the sketch of the proof of non-linear stability.

2. Preliminaries

Let $Oxyz$ be an orthogonal frame of reference with unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ ($\mathbf{k}$ pointing vertically upwards). We assume that the horizontal layer $z \in [0,d]$ is occupied by a porous medium and is rotating about the z-axis, under the actions of a vertical gravity field $\mathbf{g} = -g\mathbf{k}$ and an adverse temperature gradient β with assigned temperatures $\{T(0) = T_L, T(d) = T_U, T_L > T_U\}$. Setting $\beta = (T_L - T_U)/d$ and denoting by $m_0 = (\mathbf{v}^* = \mathbf{0}; T^* = -\beta z + T_L, p^*)$ the thermal conduction solution, in the presence of inertia the non-linear dimensionless equations governing the perturbation to m_0 may be derived from Vadasz [16], according to [24], as

$$\begin{cases} \dfrac{1}{V_a}\dfrac{\partial \mathbf{u}}{\partial t} = -\nabla\pi + R\theta\mathbf{k} + \mathcal{T}(\mathbf{u}\times\mathbf{k}) - \mathbf{u} \\ \nabla\cdot\mathbf{u} = 0 \\ \dfrac{\partial\theta}{\partial t} + \mathbf{u}\cdot\nabla\theta = Rw + \Delta\theta \end{cases} \tag{2.1}$$

where $\mathbf{u} = (u,v,w), \theta, \pi$ are the perturbations to the (seepage) velocity field, temperature field and pressure field, respectively. Moreover $V_a = \phi Prd^2/k_1$, $T = 2d^2\Omega_1/\nu$, $R^2 = \alpha g\beta d^2/kd$ are the Vadasz, Taylor and Rayleigh (dimensionless) numbers where: d is the fluid depth, k is the thermal diffusivity, ν is the viscosity, k_1 is the permeability of the medium, α is the thermal expansion coefficient, ϕ is the porosity. To system (2.1) we add the initial conditions

$$\mathbf{u}(\mathbf{x},0) = \mathbf{u}_0(\mathbf{x}), \quad \theta(\mathbf{x},0) = \theta_0(\mathbf{x}) \tag{2.2}$$

and the boundary conditions

$$\frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = w = \theta = 0 \quad \text{on } z = 0, \ z = 1. \tag{2.3}$$

In the sequel, as usual, we assume that:

(1) the perturbations $(\mathbf{u},\theta)$ are periodic in the x and y directions of periods $2\pi/a_x$, $2\pi/a_y$, respectively;
(2) $\Omega = [0, 2\pi/a_x] \times [0, 2\pi/a_y] \times [0,1]$ is the periodicity cell;
(3) $\mathbf{u}, \theta$, with their first and second spatial derivatives, are square integrable in Ω, $\forall t \in \mathbb{R}^+$ and can be expanded in Fourier series uniformly convergent in Ω.

3. Independent unknown fields

Let us consider the boundary value problem (b.v.p.)

$$\begin{cases} \dfrac{1}{V_a}\dfrac{\partial \mathbf{u}}{\partial t} = -\nabla\pi + R\theta\mathbf{k} + \mathcal{T}(\mathbf{u}\times\mathbf{k}) - \mathbf{u} \\ \nabla\cdot\mathbf{u} = 0 \\ \dfrac{\partial u}{\partial z} = \dfrac{\partial v}{\partial z} = w = \theta = 0 \quad \text{on } z = 0,1. \end{cases} \tag{3.1}$$

On taking the z-component of the curl and of the double curl of $(3.1)_1$ one obtains the following b.v.p.:

$$\begin{cases} \dfrac{\partial\zeta}{\partial t} = -V_a\zeta + \mathcal{T}V_a\dfrac{\partial w}{\partial z} \\ \dfrac{\partial\Delta w}{\partial t} = -V_a\mathcal{T}\dfrac{\partial\zeta}{\partial z} - V_a\Delta w + RV_a\Delta_1\theta \\ \dfrac{\partial\zeta}{\partial z} = w = \theta = 0 \quad \text{on } z = 0,1 \end{cases} \tag{3.2}$$

where

$$\zeta = \nabla\times\mathbf{u}\cdot\mathbf{k} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}.$$

Therefore on setting

$$Z = \frac{\partial\zeta}{\partial z} \tag{3.3}$$

one obtains

$$\begin{cases} \dfrac{\partial Z}{\partial t} = -V_a Z + \mathcal{T}V_a\dfrac{\partial^2 w}{\partial z^2} \\ \dfrac{\partial\Delta w}{\partial t} = -V_a\mathcal{T}Z - V_a\Delta w + RV_a\Delta_1\theta \\ Z = w = \theta = 0 \quad \text{on } z = 0,1. \end{cases} \tag{3.4}$$

By virtue of $\nabla\cdot\mathbf{u} = 0$, it follows that

$$\Delta_1 u = -\frac{\partial^2 w}{\partial x\partial z} - \frac{\partial\zeta}{\partial y}, \quad \Delta_1 v = -\frac{\partial^2 w}{\partial y\partial z} + \frac{\partial\zeta}{\partial x} \tag{3.5}$$

where $\Delta_1 = (\partial^2\cdot/\partial x^2) + (\partial^2\cdot/\partial y^2)$.

Let $L_2^*(\Omega)$ be the set of the functions Φ such that

(i) $\Phi : (\mathbf{x},t) \in \Omega\times\mathbb{R}^+ \to \Phi(\mathbf{x},t) \in \mathbb{R}$, Φ (together with the first derivatives and the second spatial derivatives) belongs to $L^2(\Omega)$, $\forall t \in \mathbb{R}^+$;
(ii) Φ is periodic in the x and y directions of periods $2\pi/a_x$, $2\pi/a_y$, respectively, and $[\Phi]_{z=0} = [\Phi]_{z=1} = 0$;
(iii) all the first derivatives and the second spatial derivatives of Φ can be expanded in Fourier series absolutely uniformly convergent in Ω, $\forall t \in \mathbb{R}^+$.

Since the sequence $\{\sin n\pi z\}(n = 1,2,\ldots)$ is complete orthogonal system for $L_2^*(\Omega)$, by virtue of periodicity it turns out that, $\forall\Phi \in L_2^*(\Omega)$ there exists the sequence $\{\tilde{\Phi}_n(x,y,t)\}$ such that

$$\begin{cases} \Phi = \displaystyle\sum_1^\infty \tilde{\Phi}_n(x,y,t)\sin n\pi z, & \dfrac{\partial\Phi}{\partial t} = \displaystyle\sum_1^\infty \dfrac{\partial\tilde{\Phi}_n}{\partial t}\sin n\pi z, \\ \Delta_1\Phi = -a^2\Phi, & \Delta\Phi = -\displaystyle\sum_1^\infty \xi_n\tilde{\Phi}_n \sin n\pi z, \end{cases} \tag{3.6}$$

with

$$\xi_n = a^2 + n^2\pi^2, \quad a^2 = a_x^2 + a_y^2, \quad \Delta\cdot = \Delta_1\cdot + \frac{\partial^2\cdot}{\partial z^2}, \tag{3.7}$$