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Analysis of local singular fields near the corner of a quarter-plane with mixed boundary conditions in finite plane elastostatics

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ABSTRACT

We consider a quarter-plane of compressible hyperelastic material of harmonic-type undergoing finite plane deformations. The plane is subjected to mixed (free–fixed) boundary conditions. In contrast to the analogous case from classical linear elasticity, we find that the deformation field is smooth in the vicinity of the vertex and is actually bounded at the vertex itself. In particular, the normal displacement remains positive eliminating the possibility of material interpenetration. Finally, explicit expressions for Cauchy and Piola stress distributions are obtained in the vicinity of the vertex.

1. Introduction

Stress analysis in the vicinity of corners and near points at which prescribed boundary data change type (for example, from Dirichlet to Neumann) has gained considerable attention in the literature (see, for example, [1–5]). In most of the aforementioned studies, stresses obtained within the context of the classical (linear) theory of elasticity exhibit singularities indicating either the presence of regions of high stress concentration or a break-down in the governing mathematical model. In addition, it is also found that the corresponding displacement fields suffer from oscillatory singularities which further lead to physically inadmissible phenomena such as wrinkling of free surfaces and material interpenetration. In an attempt to address the deficiencies in the classical models, more recent analyses have sought to incorporate the effects of finite strain, for example, on the corresponding singular fields near a corner and/or interface crack-tip [6,7]. In the case of mixed boundary-value problems in finite elastostatics, Knowles and Sternberg [8] have shown that for harmonic materials (see, for example, John [9], Ogden and Isherwood [10] and the references contained therein), the oscillatory singularities arising at the points on the boundary where the data change type, disappear completely.

In this paper, we continue the work of Knowles and Sternberg and consider the local finite-strain analysis in the vicinity of the corner of a quarter-plane of compressible hyperelastic material of harmonic-type undergoing finite plane deformations (see Fig. 1). The plane is subjected to mixed (free–fixed) boundary conditions. We show that, in contrast to the analogous results obtained within the linear theory of elasticity, the deformation field remains bounded at the vertex and exhibits smooth behavior in its vicinity. In particular, explicit conditions are obtained which guarantee that the corresponding normal displacements are always positive: this excludes the (unrealistic) possibility that the body can penetrate into the fixed lower half plane. With respect to local stress distributions, we note that certain components of stress are free of singularities, while others remain unbounded at the vertex. This could be explained by the sudden change in boundary data type at the corner. To analyze this further, we undertake a brief examination of the same physical problem but with fixed–fixed boundary conditions. Finally, explicit expressions for deformations and stresses (Cauchy and Piola) in the vicinity of the corner are presented.

2. Notation and prerequisites

In this section, we present the basic formulation of a harmonic material subjected to plane - strain deformations. For more details, see Knowles and Sternberg [8]. Let $z=x_1+ix_2$ be the initial coordinates of a material particle in the undeformed configuration and $w(z)=y_1(z)+iy_2(z)$, the corresponding spatial coordinates in the deformed configuration. The components of the deformation gradient tensor are given by:

$$F_{ij}=\frac{\partial y_i}{\partial x_j}=y_{i,j}, \quad i,j=1,2$$

and we define the scalar invariants as

$$I=\lambda_1+\lambda_2=\sqrt{F_{ij}F_{ij}+2J}, \quad J=\lambda_1\lambda_2=\det[F_{ij}]>0,$$

where λ_1 and λ_2 are principal stretches.

Harmonic materials proposed by John [9] are characterized by the following strain-energy density W defined per-unit-area of

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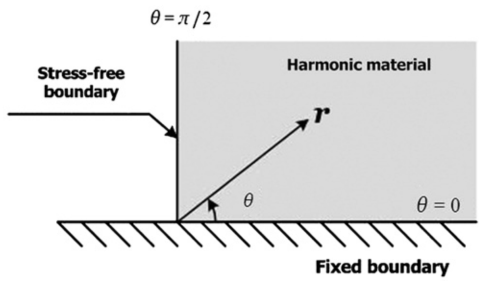

Fig. 1. Schematic of the problem.

the reference configuration:

$$W=2\mu[F(I)-J], \tag{1}$$

where μ is a positive material constant and $F(I)$ is a (scalar) material function of the invariant I. Following [8], The Piola stress and the (symmetric) actual Cauchy stress components can be expressed as follows:

$$\sigma_{12}+i\sigma_{22}=2\mu\left[\frac{F'(I)}{I}(w_{,2}+iw_{,1})-iw_{,1}\right],$$

$$\sigma_{11}+i\sigma_{21}=2\mu i\left[w_{,2}-\frac{F'(I)}{I}(w_{,2}+iw_{,1})\right], \tag{2}$$

$$\tau_{11}+\tau_{22}=2\mu\left[\frac{IF'(I)}{J}-2\right],$$

$$\tau_{11}-\tau_{22}+2i\tau_{12}=2\mu\frac{F'(I)}{IJ}(w_{,2}^2+w_{,1}^2), \tag{3}$$

where $F'(I)=dF/dI$ and I and J are given accordingly by,

$$I=|w_{,2}+iw_{,1}|,\quad J=-\mathrm{Im}|w_{,1}\overline{w_{,2}}|. \tag{4}$$

Also, the basic equilibrium equation has the following form (see Ru [11]):

$$\frac{F'(I)}{I}(w_{,2}+iw_{,1})=\Phi'(z), \tag{5}$$

where $\Phi'(z)$ is an analytic function and the overbar '–' denotes the complex conjugate. For the present local singularity analysis near the corner (where the value of I is unbounded), we follow [8] and assume that the asymptotic form of $F(I)$ is described by the expression

$$\frac{F'(I)}{I}=\alpha+\frac{\beta}{I}+F_0(I),\quad I\to\infty,\ \alpha=1,\ -1<\beta<0, \tag{6}$$

where $F_0(I)$ is a higher-order small term in the vicinity of the vertex such that

$$IF_0(I)\to 0,\quad I\to\infty.$$

Throughout the analysis we shall employ this particular asymptotic form for $F(I)$.

3. The local singular field near a corner ($\alpha=1$)

Let us now consider a quarter - plane fixed at $\theta=0$ and with stress-free boundary at $\theta=\pi/2$ (see Fig. 1). We assume that the general solution takes the following form:

$$\phi(z)=Az^\rho+Cz^\rho\ln z+\phi_0,\quad \psi(z)=Bz^\rho+Dz^\rho\ln z+\psi_0, \tag{7}$$

where ρ is an unknown real number, and A, B, C, D, ϕ_0 and ψ_0 are complex constants to be determined.

Near the vertex, where the value of I is unbounded, to second order, $F'(I)/I=1+\beta/I$. Substituting into the equilibrium equation (5) and neglecting all higher-order small terms, we obtain

$$w_{,2}+iw_{,1}=\frac{\Phi'(z)}{1+\beta/I}=\frac{\Phi'(z)(1-\beta/I)}{1+(\beta/I)^2}\approx\Phi'(z)\left(1-\frac{\beta}{I}\right). \tag{8}$$

For the present case, all leading-order singular terms are of the form $\ln r$ (see (7), independent of θ) and therefore,

$$-\frac{\beta}{I}\Phi'(z)=C+O\left(\frac{1}{\ln r}\right),$$

where C is an arbitrary constant. Consequently, to second-order approximation, the equilibrium equation (8) becomes

$$w_{,2}+iw_{,1}=\Phi'(z)+C,$$

and the general solution of w can be expressed in terms of two analytic functions $(\phi(z),\psi(z))$ as:

$$w(z)=\phi(z)+\overline{\psi(z)},\quad \phi(z)=\frac{1}{2i}\Phi(z)+\frac{Cz}{2i}. \tag{9}$$

Further, substituting the above into (5), we obtain the following expressions for I and J:

$$I=2|\phi'(z)|,\quad J=[\phi'(z)\overline{\phi'(z)}-\psi'(z)\overline{\psi'(z)}]=|\phi'(z)|^2-|\psi'(z)|^2. \tag{10}$$

Now, at the fixed boundary ($\theta=0$), we obtain, from (9), that

$$w(z)=\phi(z)+\overline{\psi(z)}=x+x_0,\quad \text{at } \theta=0, \tag{11}$$

where x_0 is an arbitrary real number. In addition, substituting the expression for $F'(I)/I$ (to second-order approximation) into the second equation of (2) yields, at the stress-free boundary ($\theta=\pi/2$):

$$\sigma_{11}+i\sigma_{21}=2\mu\left[w_{,1}-\frac{2\beta}{I}\phi'(z)\right]=0+0i,\quad at\ \theta=\pi/2. \tag{12}$$

By substituting the expressions for $\phi(z)$ and $\psi(z)$ in (7) into (11) and further writing $z=e^{i\theta}$, we obtain (at $\theta=0$)

$$(A+\overline{B})r^\rho+(C+\overline{D})r^\rho\ln\ r+\phi_0+\overline{\psi_0}=r+r_0,\quad \because r=x\quad \text{at } \theta=0. \tag{13}$$

Comparing coefficients on both sides of Eq. (13) yields

$$(C+\overline{D})r^\rho\ln r=0,\quad r=(A+\overline{B})r^\rho,\ r_0=\phi_0+\overline{\psi_0}\ \text{(Rigid body motion)}.$$

Therefore, we have that

$$C+\overline{D}=0,\quad A+\overline{B}=1,\ \rho=1,\ \phi_0+\overline{\psi_0}=0. \tag{14}$$

Further, in view of (10) and (11), the stress-free boundary condition (12) can be re-written as:

$$\phi'(z)+\overline{\psi'(z)}-\frac{2\beta}{2|\phi'(z)|}\phi'(z)=0.$$

From (7) and (14), the above becomes

$$1+i\pi C+\frac{\beta(A+C+C(\ln r+i\pi l2))}{|A+C+C(\ln r+i\pi l2)|}=0,\quad \text{at } \theta=\frac{\pi}{2}.$$

Neglecting all higher - order small terms (e.g. $A/\ln r\approx 0$) and further noting that $\ln r<0$ near the corner, we obtain from the above that

$$1+i\pi C-\frac{\beta C}{|C|}=0,$$

which determines the unknown complex constant ($C=C_1+iC_2$) in terms of β:

$$C_1=\pm\frac{\beta}{\pi}\sqrt{1-\beta^2},\quad C_2=\frac{1-\beta^2}{\pi},\ -1<\beta<0. \tag{15}$$

We mention here that, as noted in the next section, $C_1<0$ (i.e. $C_1=(-\beta/\pi)\sqrt{1-\beta^2}$) seems a reasonable choice guaranteeing $J>0$ near the corner field. Finally, the second-order local solution in the