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Wave-modulated orbits in rate-and-state friction

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ABSTRACT

A frictional spring-block system has been widely used historically as a model to display some of the features of two slabs in sliding frictional contact. Putelat et al. (2008) [7] demonstrated that equations governing the sliding of two slabs could be approximated by spring-block equations, and studied relaxation oscillations for two slabs driven by uniform relative motion at their outer surfaces, employing this approximation. The present work revisits this problem. The equations of motion are first formulated exactly, with full allowance for wave reflections. Since the sliding is restricted to be independent of position on the interface, this leads to a set of differential-difference equations in the time domain. Formal but systematic asymptotic expansions reduce the equations to differential equations. Truncation of the differential system at the lowest non-trivial order reproduces a classical spring-block system, but with a slightly different "equivalent mass" than was obtained in the earlier work. Retention of the next term gives a new system, of higher order, that contains also some explicit effects of wave reflections. The smooth periodic orbits that result from the spring-block system in the regime of instability of steady sliding are "decorated" by an oscillation whose period is related to the travel time of the waves across the slabs. The approximating differential system reproduces this effect with reasonable accuracy when the mean sliding velocity is not too far from the critical velocity for the steady state. The differential system also displays a period-doubling bifurcation as the mean sliding velocity is increased, corresponding to similar behaviour of the exact differential-difference system.

1. Introduction

Frictional spring-block systems have been studied for decades in search of a better understanding of friction. For practical and engineering interests, they also constitute good mechanical analogues of experimental apparatus, machines and tools. Moreover, they are useful "toy model" non-linear systems for studying the dynamics of frictional stick-slip oscillations [1,2] which represent a possible mechanism responsible for the recurrence of earthquakes [3]. One could, however, argue legitimately that a spring-block system cannot "correspond closely to an actual fault" [2] whose behaviour depends on continuum mechanics fields.

In this paper, we present a method for reducing the dynamics of a frictional elastic continuum to the dynamics of a sliding block pulled with a generalised Kelvin–Voigt model (a spring and a dashpot in parallel) when elastic radiation and boundary reflection are accounted for. This particular aspect concerns only the modelling of the stress waves and is independent of the model of friction that is employed to complete the formulation. We thus propose a systematic method for deriving sliding-block mechanical analogues of frictional elastic continua that are useful for the investigation of the non-linear dynamics of sliding friction and the states of erratic sliding of frictional systems, and may provide new insight into the episodic recurrence of earthquakes and aftershocks.

Earthquakes are recurrent and aperiodic, while basic stick-slip oscillations are periodic. Experimentally, irregular slip patterns have been observed at very low driving velocities for which elastic radiation is commonly disregarded [4–6]. We will show that taking into account elastic radiation allows the appearance of complex slip dynamics even for low driving velocities.

Fully developed stick-slip oscillations are relaxation oscillations that comprise a long quasi-stationary phase[1] during which the stress builds up linearly in time followed by a sudden and short harmonic slip phase accompanied by a stress drop releasing

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[1] In this paper we define the "quasi-stationary phase" as the part of a periodic orbit on which the acceleration is negligible, reserving "quasi-static" for a part of an orbit, or a system, in which elastic wave propagation is disregarded.

the elastic energy stored during the first phase [2,7]. Although Coulomb's model of friction captures the essence of stick-slip oscillations from the difference in values between the static and dynamic coefficients of friction [8], it cannot account for the existence of a velocity-dependent critical value of the stiffness for the appearance of stick-slip and the increase of stick-slip amplitude for decreasing stiffness or velocity induced by slip memory effects [9]. These experimental observations were reproduced theoretically, as we recall below, only from the concept of rate-and-state friction proposed by Ruina [4] and Rice and Ruina [10] following Dieterich [11].

Rate-and-state friction is a general framework for the quantitative description of friction laws in which the frictional shear stress τ is determined by relations of the type

$$\tau = F(v,\phi;\sigma) \quad \text{and} \quad \dot{\phi} = -G(v,\phi;\sigma), \tag{1}$$

where v and σ denote the interfacial slip rate and normal stress while ϕ represents an internal variable characterising the state of resistance to sliding of the interface. The evolution law $(1)_2$ models the memory effects typical of the response of frictional interfaces to sudden velocity changes. The instantaneous frictional response described by the law (1) implies the steady-state friction law

$$\tau = F_{ss}(V;\sigma), \tag{2}$$

obtained for slipping at constant rate $v=V$ and constant interfacial state $\phi = \phi_{ss}(V;\sigma)$ given implicitly by solving $G(V,\phi_{ss};\sigma)=0$. Accounts of the phenomenological description and geophysical applications of such laws can be found in the review articles of Marone [12] and Scholz [13], while the present state of our physical understanding of such laws and their microphysical foundations are reviewed and discussed in Baumberger and Caroli [14] and Putelat et al. [15].

Phenomenologically, the concept of rate-and-state friction assumes that a reference value of the friction coefficient associated with a reference slip rate V_* is modified by correction terms that depend on the velocity and the interfacial state. It is supposed that the interfacial state relaxes to a steady state after sliding over a length characterised by a memory length L. A common realisation of such friction laws is the Dieterich ageing law defined by

$$\tau = [a_* + a\ln(v/V_*) + b\ln(\phi/\phi_*)]\sigma \quad \text{with} \quad \dot{\phi} = 1 - v\phi/L, \tag{3}$$

where $\phi_* = L/V_*$ is the steady-state reference value of the interfacial state. Typical values for the material parameters are given in Table 1. From a microphysical point of view the memory length is usually thought to correspond to the slip distance required for the rejuvenation of the population of interacting microasperities which constitute the interface topography [11,14,16]. Besides, in the thermodynamic theory for slip events based on the Eyring transition-state theory of rate processes [15,17,18], we note that the reference slip rate V_* can be identified as the product of a reference frequency of slip events and a characteristic length corresponding to the average separation between the energy barriers to overcome in relation to some thermal activation mechanism. We finally note that the analytical form of the state evolution law is empirical and is still open to discussion (see e.g. [15]). We will use the law (3) to illustrate numerically the analyses reported in this paper.

Table 1
Typical material parameter values used in the Dieterich law (3) [12,17].

Material	a_*	a	b	L (m)	V_* (m/s)	ρ (m kg^{-3})	G (Pa)	σ (Pa)
Paper	0.369	0.0349	0.0489	0.9×10^{-6}	10^{-6}	800	10^6	10^3
Rock	0.6	0.01	0.015	20×10^{-6}	10^{-6}	2500	10^{10}	10^8

Within this rate-and-state framework, consider a block of mass M pulled with a constant speed V by a spring of stiffness k. When friction is velocity-weakening, stick-slip motion arises from a Hopf bifurcation located at a critical value k_c of the stiffness given by

$$k_c = -G_\phi F'_{ss} + M\omega_c^2, \tag{4}$$

where

$$\omega_c^2 = -G_\phi^2 F'_{ss}/F_v, \tag{5}$$

denotes the critical frequency of oscillations [5,10,17,19]. The critical stiffness and frequency depend only on the velocity-dependent frictional properties of the interface, conveyed by the slope $F'_{ss}(V) < 0$ of the steady-state friction law and the partial derivatives $F_v \equiv \partial F/\partial v$ and $G_\phi \equiv \partial G/\partial \phi > 0$ evaluated at the steady state (V,ϕ_{ss}). We note that the inertia of the block promotes positive deviations from the quasi-static value $k_* = -G_\phi F'_{ss}$ of the critical stiffness at high frequency.

In Putelat et al. [7], a first step towards connecting the dynamics of a slipping interface to the dynamics of a spring-block system was performed in the context of the problem illustrated in Fig. 1. Two horizontally infinite identical elastic slabs of thickness $h/2$ are driven in opposite directions with a uniform speed $V/2$ and slide against each other along a flat frictional interface at $z=0$ subjected to a normal stress σ. The density, the shear wave speed and the shear modulus of the slabs are denoted ρ, c_s and $\mathsf{G} = \rho c_s^2$, respectively. Assuming the interfacial slip to be uniform, the displacement in the two layers is horizontal and denoted $u(z,t)$, where z is the vertical coordinate. Assuming symmetry, it suffices to consider a velocity field in the upper layer of the form

$$\dot{u}(z,t) = V/2 + f(t-(z-h/2)/c_s) - f(t+(z-h/2)/c_s), \tag{6}$$

which accounts for shear waves radiating away from the interface and reflecting back from the top boundary. Eq. (6) implies that the interfacial slip rate $v(t) = \dot{u}(0^+,t) - \dot{u}(0^-,t)$ and the rate of interfacial shear stress $\dot{\tau}(t)$ (from the time derivative of Hooke's law $\dot{\sigma}_{xz} = \mathsf{G}\dot{u}_z$) are given by

$$\begin{cases} v = V + 2[f(t+h/(2c_s)) - f(t-h/(2c_s))], \\ \dot{\tau} = -\rho c_s[f'(t+h/(2c_s)) + f'(t-h/(2c_s))]. \end{cases} \tag{7}$$

The complete system upon which the analysis of this paper is based comprises (7) together with the interfacial friction law (1), or equivalently after differentiating (1),

$$\dot{\tau} = F_v\dot{v} - GF_\phi \quad \text{and} \quad \dot{\phi} = -G, \tag{8}$$

where the functions F_v, F_ϕ and G are evaluated at (v,ϕ,σ).

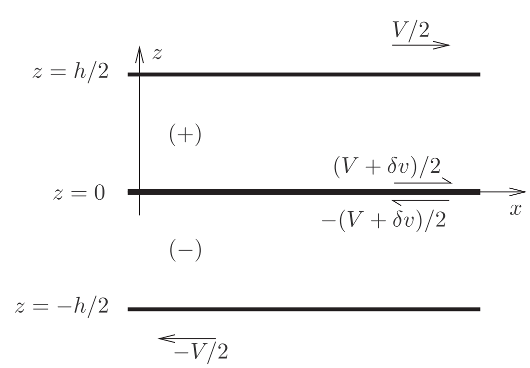

Fig. 1. A single interface system: two identical elastic slabs slide in opposite directions at constant speed $\pm V/2$; elastic shear waves radiate from the frictional interface and reflect at the boundaries $z = \pm h/2$.