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On third order density contrast expansion of the evolution equation for wrinkled unsteady premixed flames

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ABSTRACT

The dynamics of flat-on-average wrinkled flame front propagating through gaseous premixtures is considered. Leading the asymptotic expansions in powers of the burnt to unburned fractional density contrast ($0<\gamma<1$) to third order, an evolution equation (called S3) is obtained for the instantaneous front shapes. It reduces to Sivashinsky's original equation (called S1) as $\gamma \longrightarrow 0$. It also modifies a previous attempt by Sivashinsky and Clavin (called S2) to improve it. Numerical integrations of the S3 equation reveals that the new quadratic and cubic non-linearities featured at 3rd order happen to mutually compensate partially one another for realistic γ's, and are negligible at $\gamma \ll 1$. As a result, the flame shape and speed solutions to S3 nearly coincide with those of a S1/S2 type of equation, even for a 10-fold density variation ($\gamma=0.9$) and for unsteady situations, provided a *single* $O(1)$ coefficient $a(\gamma)$ be adjusted therein, *once for all* for each γ. The $O(\gamma^2)$ (and small) correction to it mainly originates from a quartic non-linearity of geometrical origin. The agreement carries over to comparisons with some DNS of 2D steady wrinkled fronts. A phenomenological (yet asymptotically correct at $\gamma \ll 1$ and exact in the linear limit) interpolating model equation is finally proposed to try and account for inertia effects associated with fast transients (e.g. acoustics related) while reproducing the above results on steady patterns.

1. Introduction

For all its connections with applications – e.g. turbulent combustion, internal combustion engines, gas turbines, industrial burners ...– the problem of wrinkled flame propagation through premixed gases is a central one in combustion science, which is by essence non-linear. Involving elliptic (for the velocity potentials) or hyperbolic (for vorticity) equations and a free boundary (the flame itself), it is further hampered by vorticity creation across the latter and by the apparent impossibility to analytically solve the Euler or Navier–Stokes equations for generic non-potential flows. Direct numerical simulations (DNS) of reactive Navier–Stokes equations may, in principle, give access to "exact results" (i.e. with no large scale modeling) on flame topology and dynamics. However, since these flames are very thin (~ 0.1 mm in usual conditions), real-scale DNS need very high spatio-temporal resolution and are extremely demanding in terms of computational resources: they are sometimes involving tens of million of CPU hours performed on hundreds of thousand of processors (see for instance recent communications [1,2]). Moreover, due to the very large amount of data to be post-processed, DNS do not necessarily allow for simple physical analysis.

On the other hand, one can take advantage of this scale separation and consider the flame as an infinitely thin front, separating the fresh (or unburned, referred to with u subscript in the sequel) mixture from the burnt (b subscript) gas. Hence, only the flame surface or curve needs be parameterized—one spatial dimension being removed. Also, many physical parameters can be lumped into few ones. The evolution equation modeling approach, consisting in finding an equation for the flame surface dynamics only – and not solving for the whole reactive flow – does not (nor claims to) replace the full 3D reactive equations. It is usually limited to simple geometrical configurations (plane, cylindrical or spherical on average). To date, no exact evolution equation is available. However, if sufficiently precise equations can be derived or built in different contexts (slow or fast transients, expanding flames, strained flames, gravity effects, acoustics...), they may provide pertinent information on flame dynamics or even be used as building blocks of (larger scale) sub grid scale modeling, e.g. in LES (large eddy simulations) of reactive flows.

In a seminal work [3], Sivashinsky realized that the unburned (ρ_u) to burnt ($\rho_b<\rho_u$) density contrast $0<\gamma \equiv (\rho_u-\rho_b)/\rho_u<1$

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may be used as a viable bifurcation parameter, albeit of a special kind since all flames have $\gamma > 0$: if $\gamma \ll 1$, the Landau–Darrieus instability mechanism of spontaneous wrinkling is weak, enabling the author of [3] to derive a leading order, weakly non-linear equation, called S1 here, and also known as Michelson–Sivashinsky equation (not to confuse it with the Kuramoto–Sivashinsky equation) for the instantaneous flame shape. Numerics [4] then analysis [5] revealed that the S1 equation describes wrinkled flames qualitatively well. An attempt to go to next (second) order in the γ expansion [6] was partly successful. Here, we correct this "S2" equation, then go to third order following the same perturbative approach. The solutions to the evolution equation (called S3) so obtained are studied numerically and in fact exhibit striking resemblance with those of an equation (called S-fit) that has the same structure as S1 (and S2, actually), provided a single γ dependent coefficient, featured in S3, be slightly modified therein. Such a resemblance carries over to comparisons about unsteady fronts and with direct simulations of steady fronts, even for realistic γ's. A phenomenological way of extending all this to fast flame-shape transients is finally proposed; it can also account for gravity effects, time-dependent or not.

The paper is organized as follows. The model and propagation are introduced in Section 2, while the coordinates and expansion scheme are presented in Section 3. The evolution Eqs. S1–S3 are derived in Section 4 (the most technical part of it being summarized in the Appendix).

Section 5 compares the solutions to S1–S3 among themselves and with others. A model for fast transients is proposed in Section 6. We end up with concluding remarks and open questions.

2. Model

At current time t, the flame is considered here to be a curve $x=F(y,t)$ in the fixed cartesian frame (x,y) defined in Fig. 1, and separates two two-dimensional incompressible flowfields $\mathbf{u}(x,y,t)$ where density ρ is either ρ_u (upstream, $x<F$) or $\rho_b=(1-\gamma)\rho_u<\rho_u$ (downstream). Euler equations are assumed to govern the velocity $\mathbf{u}=(u,v)$ and pressure p on both sides. Rankine–Hugoniot relationships, specialized to vanishingly-small Mach numbers [8], are meant to hold across the line $x=F(y,t)$. For

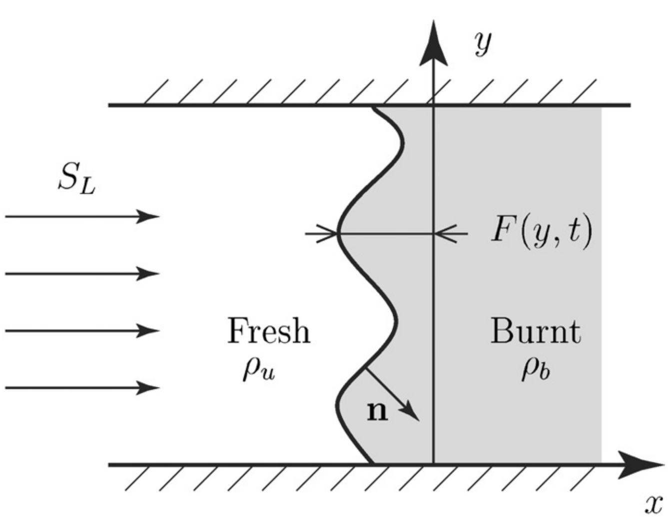

Fig. 1. Two-dimensional configuration: a plane-on-average flame front propagating along a 2-D channel. The coordinates are such that the far-upstream velocity is S_L along the x-axis, whatever $F(y,t)$.

simplicity, the Markstein [9] local propagation law is postulated

$$\mathbf{n}.(\mathbf{u}-\mathbf{D})_{x=F^-}=S_L\left(1-\frac{\gamma}{2k_n}F_{yy}\Big/(1+F_y^2)^{3/2}\right), \tag{1}$$

where $\mathbf{n}\equiv(1,-F_y)/(1+F_y^2)^{1/2}$ is the local unit normal to the front, $\mathbf{D}\equiv(F_t,0)$, the subscripts y or t represent partial differentiations (e.g. $(.)_y\equiv\partial(.)/\partial y$) and $S_L>0$ is the prescribed propagation speed of a flat flame. The prescribed $k_n>0$, to be later identified with a neutral wavenumber, is related to the effective Markstein length $(\mathcal{L})$ by $\mathcal{L}k_n=\gamma/2$, that is the only local length scale of the problem. Once endowed with appropriate lateral boundary conditions, here taken to be $L_{box}\equiv 2\pi/k_{box}$-periodicity along the y direction for some $k_{box}<k_n$, the Euler equations and Hugoniot relations are in principle enough to compute $\mathbf{u}_{x=F^-}$ in terms of a presumed – and smooth enough – $F(y,t)$.

Then, Eq. (1) explicited as

$$F_t+S_L\left(\sqrt{1+F_y^2}-1\right)+F_y v|_{x=F^-}=u|_{x=F^-}-S_L+S_L\frac{\gamma}{2k_n}\frac{F_{yy}}{1+F_y^2}, \tag{2}$$

should provide an evolution equation for $F(y,t)$ itself. Note that the speed S_T at which the front advances on average towards the uniform fresh mixture $(x=-\infty)$ is given – through overall continuity argument – by

$$1\le S_T(t)/S_L=\left\langle\sqrt{1+F_y^2}\right\rangle\simeq 1+\left\langle F_y^2/2(1-F_y^2/4+\ldots)\right\rangle \tag{3}$$

because the curvature term in (1) vanishes upon averaging along the transverse coordinate y: the operation denoted $\langle . \rangle$ is defined as

$$\langle f(y)\rangle L_{box}\equiv\int_0^{L_{box}} f(y)\,dy. \tag{4}$$

Thus, S_T/S_L-1 is the fractional increase (per unit y) in flame length caused by wrinkling. Interestingly, Eq. (3) holds whenever (1) is replaced by $\mathbf{n}.(\mathbf{u}-\mathbf{D})_{x=F^-}=S_L(1-q_y(y)/(1+F_y^2)^{1/2})$, for any $q(y)$. In particular, Eq. (3) is valid for the steady patterns considered in [10], where $q_y(y)$ also accounts for stretch. As evoked later on in (5.4), relation (3) applies to in even more general situations, including those considered in the DNS of the problem at hand.

3. Curved coordinates and small γ expansions

At this stage, it is convenient to change the streamwise coordinate from x to $X=x-F(y,t)$, then to introduce the scalings summarized in Table 1.

Such scalings, where $(U^\pm,V^\pm,P^\pm,\xi,\eta,\tau)$ are meant to be $O(1)$ as $\gamma\longrightarrow 0$, were dictated by the need of balancing F_t/S_L, $u|_{x=F^-}/S_L-1$ (estimated from the exact linearized dynamics [11,12] at $\gamma\ll 1$, see Eq. (25)), $\gamma F_{yy}/k_n$, and the leading order non-linearity $\sqrt{1+F_y^2}-1\sim F_y^2/2$, all featured in (2). As for the estimates on v and p, they follow from the continuity equation and the momentum balances, respectively. The scaling of X is

Table 1
The various non-dimensional variables (ξ,η,τ) and unknowns $(U^\pm,V^\pm,P^\pm,\phi)$; the latter are $O(1)$ but all vanish for a steady, flat flame.

	Fresh side $(X<0)$	Burnt side $(X>0)$
$k_nX=\xi$	$u=S_L(1+\gamma^2U^-)$	$u=S_L\left(\frac{1}{1-\gamma}+\gamma^2U^+\right)$
$k_ny=\eta$	$v=S_L\gamma^2V^-$	$v=S_L\gamma^2V^+$
$\gamma S_Lk_nt=\tau$	$p=p_u+\rho_uS_L^2\gamma^2P^-$	$p=p_b+\rho_uS_L^2\gamma^2P^+$
$k_nF=\gamma\phi$	$p_b-p_u=-\rho_uS_L^2\frac{\gamma}{1-\gamma}$	