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Generating functions for volume-preserving transformations

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ABSTRACT

A general implicit solution for determining volume-preserving transformations in the n-dimensional Euclidean space is obtained in terms of a set of $2n$ generating functions in mixed coordinates. For $n=2$, the proposed representation corresponds to the classical definition of a potential stream function in a canonical transformation. For $n=3$, the given solution defines a more general class of isochoric transformations, when compared to existing methods based on multiple potentials. Illustrative examples are discussed both in rectangular and in cylindrical coordinates for applications in mechanical problems of incompressible continua. Solving exactly the incompressibility constraint, the proposed representation method is suitable for determining three-dimensional isochoric perturbations to be used in bifurcation theory. Applications in non-linear elasticity are envisaged for determining the occurrence of complex instability patterns for soft elastic materials.

1. Introduction

Considering a bounded region Ω_0 in the n-dimensional Euclidean space, this work is aimed at defining generating functions for volume-preserving transformations of a set of continuously differentiable functions $u_j = u_j(U_1,U_2,\ldots,U_n) : \Omega_0 \to \Re$, with $j=1,2..,n$. Such an isochoric constraint can be expressed by a non-linear first-order partial differential equation as follows:

$$J(U_1,U_2,\ldots,U_n) = \det \frac{\partial(u_1,u_2,\ldots,u_n)}{\partial(U_1,U_2,\ldots,U_n)} = 1 \tag{1}$$

where J is defined as the Jacobian of the transformation. The cases $n=2,3$ are of particular interests in continuum mechanics, because the functions U_j,u_j can be treated as the material/spatial components of the position vectors $\mathbf{U}$, $\mathbf{u}=\mathbf{u}(\mathbf{U})$ in the reference/actual configuration, respectively. In such a case, the Jacobian defined in Eq. (1) corresponds to the determinant of the deformation tensor $\mathbf{F} = \text{Grad}\,\mathbf{u} = \partial\mathbf{u}/\partial\mathbf{U}$, so that the functions u_j determine the deformation fields for an incompressible material. For $n=2$, the solution of Eq. (1) corresponds to an area-preserving transformation, as reported by Bateman [1], who ascribed its first formulation to Gauss. Rooney and Carroll [7] realized that such a solution could be expressed by an implicit representation through the definition of a stream function. Using this change of notation, the governing equations have the structure of Hamilton's canonical equations with one degree of freedom, therefore such a stream function can be regarded as a generating function for a canonical transform of planar coordinates. The extension of this solution to $n \geq 3$ was considered by Carroll [4], who proposed an implicit representation by the means of $(n-1)$ potential functions, restricted by a set of $(n-1)$ admissibility conditions. Another implicit solution was later proposed by Knops [6], transforming the problem to a linear first-order non-homogeneous differential equation by using prescribed cofactors in the expanded expression for the Jacobian, recovering Carroll's expression for $n=3$. Although representing complete solutions of the differential problem given by Eq. (1), both methods are given in implicit form and their application might be difficult for seeking explicit solutions with given boundary conditions imposed by the mechanical problem under consideration.

This work is organized as follows. In Section 2, the existing description of volume-preserving transformation using coupled potential functions is analyzed, underlying its limitations for continuum mechanics applications. In Section 3, the definition of generating functions for volume-preserving transformation is given for a general n-dimensional problem. The three-dimensional case is particularly examined, highlighting possible applications for stability problems in non-linear elasticity. The results are finally summarized in Section 4.

2. Limitations of existing solutions

In this paragraph, the solution for a generic isochoric deformation presented by Carroll [4] is analyzed. Choosing $n=3$ for the sake of simplicity, the volume-preserving transformation is given in terms of two potential functions $\Phi(X,y,z)$ and $\psi(X,Y,z)$, referring to different mixed coordinate systems. The general solution takes the following implicit form:

$$x = \frac{\partial \Phi(X,y,z)}{\partial y} \tag{2}$$

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$$Z = \frac{\partial \psi(X,Y,z)}{\partial Y} \tag{3}$$

$$\frac{\partial \Phi(X,y,z)}{\partial X} = \frac{\partial \psi(X,Y,z)}{\partial z} \tag{4}$$

In order to understand if the solution given by Eqs. (2)–(4) is able to represent a generic isochoric deformation, the multiplicative decomposition $\mathbf{F}=\mathbf{F}_1\mathbf{F}_2\mathbf{F}_3$ is introduced, representing the local changes of coordinates sketched in Fig. 1.

It is straightforward to show that the local deformation gradients between the mixed coordinate states can be expressed as

$$\mathbf{F}_1 = \begin{bmatrix} \Phi_{,yX} & \Phi_{,yy} & \Phi_{,yz} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{F}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{\psi_{,YX}}{\psi_{,Yz}} & -\frac{\psi_{,YY}}{\psi_{,Yz}} & \frac{1}{\psi_{,Yz}} \end{bmatrix} \tag{5}$$

where comma denotes partial differentiation, and the admissibility condition $\psi_{,Yz} \neq 0$ is set to avoid local singularities. Similarly, the tensor $\mathbf{F}_2$ can be given with respect to $y=y(X,Y,z)$ and $Y=Y(X,y,z)$, as follows:

$$\mathbf{F}_2 = \begin{bmatrix} 1 & 0 & 0 \\ y_{,X} & y_{,Y} & y_{,z} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{Y_{,X}}{Y_{,y}} & \frac{1}{Y_{,y}} & -\frac{Y_{,z}}{Y_{,y}} \\ 0 & 0 & 1 \end{bmatrix} \tag{6}$$

The incompressibility condition for the overall deformation can be derived using Eqs. (5) and (6) in the following form:

$$\det \mathbf{F} = \frac{\Phi_{,yX}}{\psi_{,zY} \cdot Y_{,y}} = \frac{\Phi_{,yX} \cdot y_{,Y}}{\psi_{,zY}} = 1 \tag{7}$$

which is identically satisfied imposing the condition in Eq. (4), together with the implicit representation given by Eqs. (2) and (3).

Using simple differentiation on both sides of Eq. (4) with respect to Z, the following identity also holds:

$$\Phi_{,yX}\frac{\partial y}{\partial Z} = (\psi_{,zz} - \Phi_{,Xz})\frac{\partial z}{\partial Z} = 0 \tag{8}$$

which reveals that the volume-preserving transformation in the solution given by Carroll [4] imposes $\partial y/\partial Z = 0$, being limited to a particular deformation field. Moreover, such an implicit representation is unable to derive explicitly the expression of the transformation of the y coordinate, limiting its practical utility for finding explicit solutions in continuum mechanics problems. In the following, the use of generating functions is investigated to define a generic n-dimensional isochoric transformation.

3. Definition of generating functions for volume-preserving transformations

In classical mechanics, canonical transformations are used in order to preserve area changes in the displacements fields, based on the definition of generating functions of mixed (one material, one spatial) coordinates which allow to define implicit relations between coordinates belonging to the same framework [8]. In the following, the definition of generating functions is given for generic volume-preserving transformations, first for the three-dimensional case and, secondly, for a general n-dimensional problem.

3.1. Isochoric displacement fields in rectangular coordinates

The definition of volume-preserving transformations using a three-dimensional generating function is investigated in the following. Dealing with a generic three-dimensional deformation in rectangular coordinates, one can try to extend the classical methodology using a mixed coordinate state (X,Y,z), so that a multiplicative decomposition $\mathbf{F}=\mathbf{F}_a\mathbf{F}_b$ can be imposed, as depicted in Fig. 2.

Assuming the existence of a generating function $f(X,Y,z)$, the following implicit relations between coordinates are defined as

$$x = \frac{\partial^2 f(X,Y,z)}{\partial Y \partial z} \tag{9}$$

$$y = \frac{\partial^2 f(X,Y,z)}{\partial X \partial z} \tag{10}$$

where the expression of $Z=Z(X,Y,z)$ has to be determined from the incompressibility constraint. According to the implicit representation given by Eqs. (9) and (10), the local deformation tensors can be expressed as follows:

$$\mathbf{F}_a = \begin{bmatrix} f_{,XYz} & f_{,YYz} & f_{,Yzz} \\ f_{,XXz} & f_{,XYz} & f_{,Xzz} \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{F}_b = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{Z_{,X}}{Z_{,z}} & -\frac{Z_{,Y}}{Z_{,z}} & \frac{1}{Z_{,z}} \end{bmatrix} \tag{11}$$

Looking for isochoric solutions of the differential problem, the incompressibility condition $\det \mathbf{F}=1$ is fulfilled by choosing the following implicit representation for the Z coordinate:

$$Z = \int^z (f^2_{,XY\eta}(X,Y,\eta) - f_{,XX\eta}(X,Y,\eta)\cdot f_{,YY\eta}(X,Y,\eta))\,d\eta + g(X,Y) \tag{12}$$

where g is an arbitrary function, and we must set $\partial Z/\partial z \neq 0$ in order to avoid local singularities.

Looking for applications in continuum mechanics, an illustrative example is given by using the following expression for the generating function:

$$f(X,Y,z) = XYz + \varepsilon \cdot h(z)\sin(k_x X)\sin(k_y Y) \tag{13}$$

where $h(z)$ is a generic function of z. Using the implicit coordinate transformations in Eqs. (9), (10), and (12) and considering ε as a small parameter, the displacements fields are defined at first

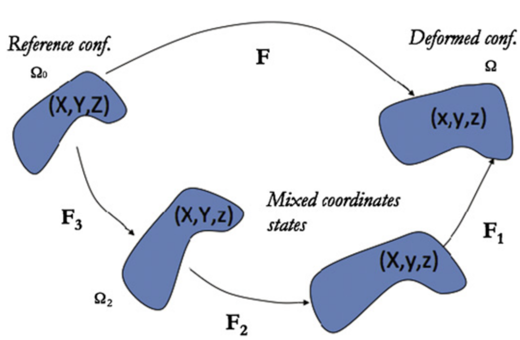

Fig. 1. Multiplicative decomposition of the deformation gradient between the reference (X,Y,Z) and the actual (x,y,z) configurations, considering two intermediate states defined in mixed coordinates as (X,Y,z) and (X,y,z).

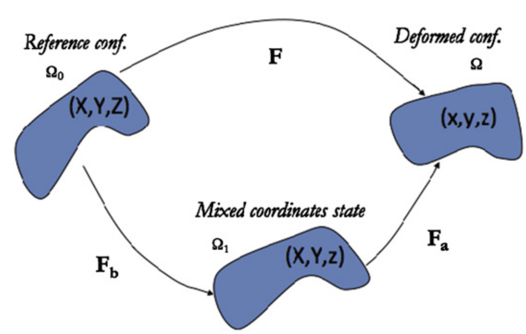

Fig. 2. Multiplicative decomposition of the deformation gradient between the reference (X,Y,Z) and the actual (x,y,z) configurations, considering an intermediate state (X,Y,z) defined in mixed coordinates.