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Exact elastic solution of the axisymmetric and deviatoric loaded hollow sphere

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ABSTRACT

This paper provides the analytical solution of the elastic hollow sphere subjected to axisymmetric and pure deviatoric surface tractions within the framework of the infinitesimal strains. The expressions of the stress and displacement fields are derived in closed-form in terms of spherical harmonics by using Boussinesq-Neuber-Papkovitch potentials. The obtained solution is valid for thin and thick hollow sphere. It is shown that, for the J_2-plasticity, the hollow spherical shell undergoes incipient first plastic strains at the pole $\theta = \pi/2$ located on the internal surface boundary. In the perspective of shakedown analysis of ductile porous materials, the macroscopic stress and strain fields of the hollow sphere model are obtained from their local counterparts by the volume average operator.

1. Introduction

Over the past few years, there has been a growing interest in Metallic Hollow Sphere Structures (MHSS) due to their light weight and their capacity for energy absorption and heat insulation. These novel foams, composed of an assemblage of hollow spheres, are used in transport engines, aerospace and chemistry [1–3]. For instance, the hollow silicalite spheres are used for ethanol/water separation by pervaporation [4], the hollow spheres ceramics are employed for heat insulation [5], the synthetic magnetic polymeric microsphere can be used for selective enrichment and rapid separation of phosphopeptides [6], etc. New technologies for manufacturing hollow spherical-cell foams with high precision for wide ranges of thickness and diameter have been developed recently [7,8]. In parallel, great efforts have been focused over characterization of different failure modes of these cellular materials which requires a better understanding of deformations of a single hollow sphere.

On the other hand, the hollow sphere shell plays a key role in ductile damage of porous materials. In fact, since the Gurson's pioneering work [9] on ductile damage of voided solids, the unit cell modeling the representative elementary volume (REV) in microporomechanics is almost the hollow sphere [10–15] because it is the simplest geometrical model. Further, it allows the derivation of closed-form expressions of the effective yield criteria according to the local strength yield of the solid matrix. In our recent works [16,17], we have adopted the hollow sphere unit cell for the shakedown study of porous materials under cyclic loads. The present work stems from these studies.

Although numerous experimental investigations and numerical studies based on the finite element method have been devoted to the hollow sphere under different loads [18–27], few analytic solutions dealing with the mechanical response to complex and general loads are provided in literature. The analytical elastic plastic solution of hollow sphere under internal and external pressure is classical and can be found in any textbook of mechanics of deformable solids. Wei et al. [28] provided the closed form expressions of the stress and strain distributions within an elastic thin or thick hollow sphere subjected to diametrical point loads. The method of solution is based on Fourier-Legendre expansion for the boundary applied loads. Motivated by Wei et al. solution, Chen et al. [29] have solved the problem of an elastic hollow sphere compressed between two flat platens under the Hertzian contact assumption. Gregory et al. [30] have developed approximate solution for a thin or a moderately thick spherical cap in axisymmetric deformations. An asymptotic expansion in the framework of the thin shell theory with refined boundary conditions has been employed. Later on, in Ref. [31], a similar procedure has been provided for the derivation of an asymptotic solution of a thick hollow sphere compressed by equal and opposite concentrated axial loads. The dynamic response of a thick-walled elastic spherical shell subject to radially symmetric loadings have been studied in Pao et al. [32] by applying the theory of rays.

The purpose of the present work is to derive the analytical solution of a hollow sphere made up of a homogeneous and isotropic material in infinitesimal elasticity under axisymmetric and deviatoric surface tractions. The plan of the paper is as follows. In the next section, a brief review of the internal solution of a solid sphere and the external

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solution of the spherical cavity embedded in infinite matrix under axisymmetric loads is presented. The presentation follows the one in Refs. [33] and [34]. In section 3, we firstly setup the problem of a hollow sphere subjected to an axisymmetic and pure deviatoric surface stress distribution on the outer boundary. Then, the closed-form expressions of the displacement and stress field are derived by the combination of the internal and external solutions. The macroscopic stress and strain fields obtained from their local counterparts by the volume average operator are also delivered. Section 4 focuses on von Mises yield condition of the hollow sphere. It is worthy to note that some computations have been checked or performed by making use of the Mathematica software [35,36]. Finally, some concluding remarks are drawn in the last section.

2. The external and internal problems

Boussinesq-Neuber-Papkovitch potentials provide a powerful tool for solving three-dimensional elastic problems. The displacement and stress fields are expressed in terms of harmonic potentials, given by the vector $\boldsymbol{\Psi}$ and the scalar function Φ.

In the absence of volume forces, the displacement field $\boldsymbol{u}$ reads:

$$2\mu\boldsymbol{u} = -(4-\nu)\boldsymbol{\Psi} + \nabla(\boldsymbol{x}\cdot\boldsymbol{\Psi}+\Phi) \tag{1}$$

where $\boldsymbol{x}$ is the position vector, μ is the shear modulus and ν is Poisson ratio.

The harmonicity of the potentials ($\nabla^2\boldsymbol{\Psi} = 0$; $\nabla^2\,\Phi = 0$) insures that Lamé-Navier equations of the linear elasticity are satisfied.

2.1. Spherical harmonics

Consider the spherical coordinates (r, θ, ϕ) where r is the radius, θ the inclination angle, φ the azimuth one, with orthonormal frame $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi\}$ as shown in Fig. 1.

For an axisymmetric potential F (independent of ϕ), the Laplace equation writes:

$$\nabla^2F(r,\theta) = \frac{\partial^2 F}{\partial r^2}(r,\theta) + \frac{2}{r}\frac{\partial F}{\partial r}(r,\theta) + \frac{cot\theta}{r^2}\frac{\partial F}{\partial\theta}(r,\theta) = 0 \tag{2}$$

Employing the method of separation of variables, the solution of (2) is decomposed in terms of Fourier series with respect of the variable θ as follows:

$$F(r,\theta) = \sum_{n=0}^{\infty} r^n f_n(\theta) \tag{3}$$

Substitution of (3) in (2) yields the following differential equation

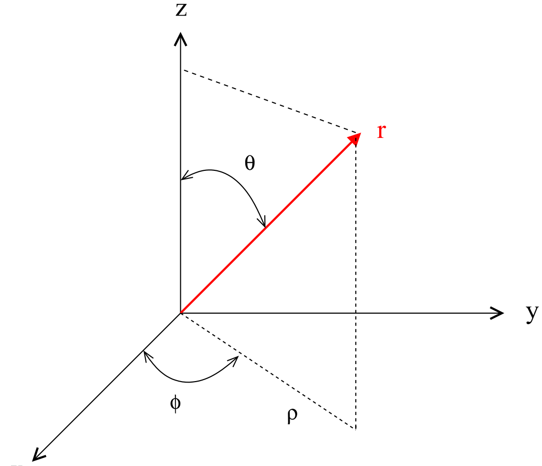

Fig. 1. The spherical coordinates.

$$\frac{1}{sin\theta}\frac{d}{d\theta}\left(sin\theta\frac{df_n}{d\theta}\right) + n(n+1)f_n = 0 \tag{4}$$

By introducing the new variable $\zeta = cos\theta$, equation (4) recasts into:

$$(1-\zeta^2)\frac{d^2 f_n}{d\zeta^2} - 2\frac{df_n}{d\zeta} + n(n+1)f_n = 0 \tag{5}$$

This is the standard Legendre equation for which the continuous fundamental solutions for $|\zeta| \le 1$ (or $0 \le \theta \le \pi$) are the Legendre polynomials of the first kind $P_n(\zeta)$:

$$P_n(\zeta) = \frac{1}{2^n n!}\frac{d^n(\zeta^2-1)}{d\zeta^n} \tag{6}$$

The solutions $r^nP_n(cos\theta)$ $(n \ge 0)$ of the Laplace equation are called the spherical harmonics.

It is worth noting that the harmonics defined by the Legendre Polynomials given by (6) are bounded at the origin $r = 0$. However, if we set $n = -p-1$ for $p \ge 0$ then $n(n+1) = p(p+1)$, and thus the Legendre equation (4) remains unchanged but with p replacing $\boldsymbol{n}$. This means that the potentials $r^{-n-1}P_n(cos\theta)$ are harmonics and also singular at the origin $r = 0$ for $n \ge 0$.

2.2. Internal solution

The internal problem is concerned with a solid sphere subjected to an axisymmetric traction exerted onto its boundary S_e defined by $r = b$. Let $\boldsymbol{u} = (u_r, u_\theta, 0)$ be the displacement vector explained in the spherical frame. It is shown in Refs. [33,34], that by using Boussinesq-Neuber-Papkovich solution (1), the displacement components are given in terms of Legendre's polynomial series as follows:

$$u_r = \sum_n \left[A_n(n+1)(n-2+4\nu)\,r^{n+1} + B_n n\,r^{n-1}\right] P_n(\zeta) \tag{7}$$

$$u_\theta = \sum_n \left[A_n(n+1)(n+5-4\nu)\,r^{n+1} + B_n\,r^{n-1}\right]\frac{d}{d\theta}P_n(\zeta) \tag{8}$$

The stresses are given by:

$$\frac{1}{2\mu}\sigma_{rr} = \sum_n \left[A_n(n+1)(n^2-n-2-2\nu)\,r^n + B_n n(n-1)\,r^{n-2}\right] P_n(\zeta) \tag{9}$$

$$\frac{1}{2\mu}\sigma_{r\theta} = \sum_n \left[A_n(n^2+2n-1+2\nu)\;r^n + B_n(n-1)\,r^{n-2}\right]\frac{d}{d\theta}P_n(\zeta) \tag{10}$$

$$\begin{aligned}\tfrac{1}{2\mu}\sigma_{\theta\theta} = &-\textstyle\sum_n \left[A_n(n^2+4n+2+2\nu)(n+1)\,r^n + B_n n^2\,r^{n-2}\right] P_n(\zeta)\\ &+\textstyle\sum_n \left[A_n(n+5-4\nu)\,r^n + B_n\,r^{n-2}\right] cot(\theta)\,\tfrac{d}{d\theta}P_n(\zeta)\end{aligned} \tag{11}$$

$$\begin{aligned}\tfrac{1}{2\mu}\sigma_{\varphi\varphi} = &\textstyle\sum_n \left[A_n(n+1)(n-2-2\nu-4n\nu)\,r^n + B_n n\,r^{n-2}\right] P_n(\zeta)\\ &+\textstyle\sum_n \left[A_n(n+5-4\nu)\,r^n + B_n\,r^{n-2}\right] cot(\theta)\,\tfrac{d}{d\theta}P_n(\zeta)\end{aligned} \tag{12}$$

Let $\boldsymbol{T} = (\sigma_{rr}(r=b), \sigma_{r\theta}(r=b), 0)$ be the stress vector applied onto the boundary S_e: $r = b$. For convenience, let us denote $f(\theta) = \sigma_{rr}(r=b,\theta)$ and $g(\theta) = \sigma_{r\theta}(r=b,\theta)$.

It follows from equations (9) and (10) that:

$$\begin{aligned}f(\theta) &= 2\mu\textstyle\sum_n \left[A_n(n+1)(n^2-n-2-2\nu)\,b^n + B_n n(n-1)\,b^{n-2}\right] P_n(\zeta)\\ &= \textstyle\sum_n F_n\,P_n(\zeta)\end{aligned} \tag{13}$$

$$\begin{aligned}g(\theta) &= 2\mu\textstyle\sum_n \left[A_n(n^2+2n-1+2\nu)\;b^n + B_n(n-1)\,b^{n-2}\right]\tfrac{d}{d\theta}P_n(\zeta)\\ &= \textstyle\sum_n G_n\,\tfrac{d}{d\theta}P_n(\zeta) = -\textstyle\sum_n G_n\,\tfrac{dP_n(\zeta)}{d\zeta}sin\theta\end{aligned} \tag{14}$$