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An analytical solution of multi-layered thick-walled tubes in thermo-elasticity with application to gas-wells

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ABSTRACT

Gas-wells are composed of several layers of different materials and undergo at least axial loads, internal and external pressure and temperature differences from inside to outside. In this article, an analytical solution for thick-walled tubes under internal and external pressure, constant axial deformation and a stationary heat distribution is provided. Here, we restrict ourselves to the case of small strain thermo-elasticity. As an example, a gas-well with several layers in the production region is investigated, where the temperature distribution can vary due to temperature cycles. The solution can also be used as verification example for finite element simulations or for comparing it to approximate solutions.

1. Introduction

Strongly fluctuating energy production using wind and solar power plants requires the storage of energy during times when the production is too large. Thus, underground storages can be used to store, for example, hydrogen in geological sub-surfaces. In this case, electrical energy is transformed into chemical energy, i.e. the surplus electrical energy is utilized to separate water into hydrogen and oxygen by means of electrolysis, [24]. The hydrogen is compressed and injected into a geological formation. During periods of less power production, but high demand, the hydrogen is extracted and transformed into electrical energy by generators or fuel cells. In this case, the completion of gas-wells is mechanically and thermally treated. The completion itself is composed of several layers of steel and concrete depending on the region. Thus, the estimation of the stress state is of particular interest.

To determine the principal stress state either finite element simulations or hand-calculation can be chosen. In this paper, we provide analytical equations for the problem under consideration (we arrive at systems of linear equations, which can principally solved analytically by means of Mathematica [31]. However, these equations are very complex for more than two layers so that “analytic” has to be understood as “exact”). Of course, we can draw on finite elements for computing the axisymmetric coupled problem either thermo-elastic or thermo-viscoplastic in order to estimate the internal stress state. However, an analytic approach has the advantage of both obtaining an insight of the distribution of the stresses, deformation and temperature as well as providing a verification example for numerical methods (code verification) or for other approximate solutions. Furthermore, an analytical equation is much easier to be evaluated. The analytical equation of the pure mechanical part is given, for example, in Ref. [17]. There, the assumption of small strains, and linear, isotropic elasticity is considered. The extension to large strains, which leads to a numerical solution, is studied in Ref. [34]. For the case of several layers of distinct elastic layers, we refer to [4,8,10,25,32]. The case of an additional axial deformation is summarized in Ref. [14]. In shrink fitting with three layers and without temperature changes, see Ref. [20]. There are also books treating pressure vessels, i.e. tubes on the subject of interest [5,12,30], p.657; [35], p.683, and [27], pp.315ff.

Further questions are, for example, different non-axisymmetric stress states and curved wells, where plane stress conditions in the rock formation are chosen to develop approximate analytical solutions, see Ref. [33]. Several (drilling, production, installation) phases are investigated in Ref. [11], which draw on analytical approximate results of [9] and [13]; but omitting the temperature differences in the rock formation and the internal gas as well. Similarly to [18,33] applied unequal horizontal stresses, but treated the temperature influence by an approximation – constant over the wall thickness. To provide an analytical approximation in situations with thin-walled wells, in particular, the casing itself [26], developed a model with the assumptions of vanishing axial strains $\varepsilon_z \approx 0$, Barlow's formulas of thin-walled tubes, and a constant mean temperature increase. These assumptions are compared to finite element simulations. Further investigations have been provided for these high temperature and high pressure conditions by Refs. [2,3,28].

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Further interest are finite element simulations, where also inelastic material properties of both the steel and the cement layers are taken into account, however, yielding to pure numerical investigations, see, for instance [1], and the references cited therein. An exact solution – in the sense of an arbitrary accurate computation – for one-layer using finite strain, von Mises plasticity is provided in Ref. [7]. Even interface failure between steel and cement is of certain importance, see, for example, [36].

The thermo-mechanically coupled problem of small strain analysis can be divided into the stationary and the transient case. The transient case is provided in Ref. [21], whereas the stationary problem of one-sided coupling is summarized here. One-sided coupling implies that the stress state is coupled with the temperature, and the heat equation is independent of the deformation. For the case of fully coupling and transient processes using finite elements, see Ref. [23]. However, it is not known to the authors that the (analytical) exact, coupled, stationary, axisymmetric thermo-elastic problem under internal and external pressures and temperatures for several layers is provided in the literature, which is the main goal of this presentation.

The structure of the article is as follows: first, the case of thermo-elasticity and the resulting balance of linear momentum for the case of axisymmetry is recapped. Additionally, the required temperature is provided by the stationary heat equations, which is a one-sided coupling problem. The equations for multiple layers of different materials is derived under the assumption of a constant deformation and temperature in axial direction. Finally, we treat a completion above and below the packer with two and four layers of a representative gas-well. The solutions are compared with finite element solutions.

The notation in use is defined in the following manner: geometrical vectors are symbolized by $\vec{a}$ and second-order tensors $\mathbf{A}$ by bold-faced Roman letters. Furthermore, we introduce matrices and column vectors symbolized by bold-faced italic letters $\boldsymbol{A}$.

2. Basic equations of thermo-elasticity in the axisymmetric case

In the case of thermo-elasticity, we have to solve the equilibrium conditions (here, without specific body-forces)

$$\operatorname{div} \mathbf{T}(\vec{x}) = \vec{0} \tag{1}$$

and the stationary heat equation

$$\kappa_\Theta \operatorname{div}(\operatorname{grad}\Theta(\vec{x})) = 0, \tag{2}$$

where $\mathbf{T}$ represents the stress tensor, $\vec{x}$ the position of the material point, κ_Θ the heat conductivity, and Θ the absolute temperature. Here, we draw on the differential operators

$$\operatorname{grad}\Theta = \nabla\,\Theta = \Theta,_k\,\vec{g}^k, \quad \operatorname{grad}\vec{u} = \vec{u}\otimes\nabla = \vec{u},_k\otimes\vec{g}^k = u^m|_k\,\vec{g}_m \otimes\vec{g}^k, \tag{3}$$

$$\operatorname{div}\vec{u} = \nabla\cdot\vec{u} = \vec{u},_k\cdot\vec{g}^k = u^m|_m, \quad \operatorname{div}\mathbf{T} = \mathbf{T}\nabla = \mathbf{T},_k\,\vec{g}^k = T_k^{\ m}|_m\,\vec{g}^k, \tag{4}$$

with the tangent and gradient vectors $\vec{g}_m$ and $\vec{g}^k$, respectively. ∇ represents the Nabla-operator, and both $u^k|_m = u^k,_m + \Gamma^k_{jm}u^j$ as well as $T_k^{\ m}|_l = T_k^{\ m},_l - \Gamma^i_{kl}T_i^{\ m} + \Gamma^m_{ik}T_m^{\ i}$ covariant derivatives, see, for example, [16,19]. These equations are accompanied by Dirichlet and Neumann boundary conditions

$$\vec{u}(\vec{x}) = \overline{\vec{u}}(\vec{x}) \text{ on } A^u \quad\text{and}\quad \vec{t}(\vec{x}) = \overline{\vec{t}}(\vec{x}) \text{ on } A^\sigma, \tag{5}$$

$$\Theta(\vec{x}) = \overline{\Theta}(\vec{x}) \text{ on } A^\Theta \quad\text{and}\quad q(\vec{x}) = \overline{q}(\vec{x}) \text{ on } A^q, \tag{6}$$

with the surface regions A^u and A^Θ, where the displacements and the temperatures are prescribed, and the boundary A^σ and A^q, where the stresses and heat flux are known. Here, $A^u \cup A^\sigma = A$ and $A^\Theta \cup A^q = A$ have to hold. Of course, Eq. (5) has to be considered in particular directions, i.e. component-wise. Thus, it is more a symbolical notation. $\vec{t} = \mathbf{T}\vec{n}$ are the tractions and $q = -\vec{q}\cdot\vec{n}$ the heat flux, where $\vec{n}$ denotes the surface normal.

Additionally to the boundary-value problem (1) and (2) we have constitutive equations defining the stress state. We start with the decomposition of the (linear) strain tensor

$$\mathbf{E}(\vec{x}, t) = \frac{1}{2}(\operatorname{grad}\vec{u}(\vec{x}, t) + \operatorname{grad}^T\vec{u}(\vec{x}, t)), \tag{7}$$

where $\vec{u}(\vec{x}, t)$ defines the displacement field, into a mechanical $\mathbf{E}_\mathrm{M}$ and a thermal part $\mathbf{E}_\Theta$,

$$\mathbf{E} = \mathbf{E}_\mathrm{M} + \mathbf{E}_\Theta. \tag{8}$$

The thermal part is purely volumetric

$$\mathbf{E}_\Theta = \alpha_\Theta\vartheta(\vec{x})\mathbf{I}, \tag{9}$$

where

$$\vartheta(\vec{x}) = \Theta(\vec{x}) - \Theta_0 \tag{10}$$

defines the temperature change, α_Θ the heat expansion coefficient, and Θ_0 the reference temperature. In other words, the mechanical strains are defined by the difference of the total strains (7) and the thermal strains (9), see Refs. [6,15]. The stress state depends on the mechanical strains

$$\mathbf{T} = \rho\frac{\mathrm{d}\psi_\mathrm{M}}{\mathrm{d}\mathbf{E}_\mathrm{M}} = K(\operatorname{tr}\mathbf{E}_\mathrm{M})\mathbf{I} + 2G\mathbf{E}_\mathrm{M}^D, \tag{11}$$

$\psi_\mathrm{M}(\mathbf{E}_\mathrm{M}) = K(\operatorname{tr}\mathbf{E}_\mathrm{M})^2/2 + G\mathbf{E}_\mathrm{M}^D\cdot\mathbf{E}_\mathrm{M}^D$ represents the specific free-energy of the mechanical strains, leading with Eq. (9) to

$$\mathbf{T} = K(\operatorname{tr}\mathbf{E} - 3\alpha_\Theta\vartheta)\mathbf{I} + 2G\mathbf{E}^D. \tag{12}$$

$K = E/(3(1-2\nu))$ defines the bulk modulus, $G = E/(2(1+\nu))$ the shear modulus, where E symbolizes Young's modulus and ν the Poisson number. $\operatorname{tr}\mathbf{E} = \mathbf{E}\cdot\mathbf{I} = E_k^{\ k}$ denotes the trace operator, and $\mathbf{E}^D = \mathbf{E} - (\operatorname{tr}\mathbf{E})/3\mathbf{I}$ the trace operator. $\mathbf{I} = \vec{g}_k\otimes\vec{g}^k$ denotes the second order identity tensor, $\mathbf{I}\vec{a} = \vec{a}$, and $\mathbf{A}\cdot\mathbf{B} = A_k^{\ l}B^k_{\ l}$ symbolizes the scalar product of two second order tensors. In conclusion, Eq. (1) depends on the displacements $\vec{u}(\vec{x})$ and the temperature $\Theta(\vec{x})$, whereas Eq. (2) is only built on $\Theta(\vec{x})$. In this respect, we can solve Eq. (2) in a first step and insert the result into Eq. (1).

2.1. Axisymmetric heat problem with several layers

Since we are interested in axisymmetric applications, Eq. (2) reduces in cylindrical coordinates (r, φ, z) to

$$\frac{1}{r}(r\Theta,_r),_r + \frac{1}{r^2}\Theta,_{\varphi\varphi} + \Theta,_{zz} = 0, \tag{13}$$

see, for example [19], where the commas symbolize the partial derivatives with respect to one of the coordinates (e.g. $\Theta,_r = \partial\Theta/\partial r$). Under the assumption of a constant temperature in circumferential direction and a constant temperature distribution in axial direction, Eq. (13) degenerates to

$$\Theta''(r) + \frac{1}{r}\Theta'(r) = 0. \tag{14}$$

Using the dimensionless parameter $\xi = r/R$, we choose $R := r_o$, the ordinary differential equation (14) with $\Theta(r) = \hat{\Theta}(\hat{\xi}(r))$, i.e.

$$\Theta'(r) = \hat{\Theta}'(\xi)\frac{d\hat{\xi}}{dr} = \frac{1}{R}\hat{\Theta}'(\xi), \quad \Theta''(r) = \frac{1}{R^2}\hat{\Theta}''(\xi),$$

leads to

$$\hat{\Theta}''(\xi) + \frac{1}{\xi}\hat{\Theta}'(\xi) = 0 \tag{15}$$

having the solution