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On the influence of failure on the dynamics of mechanical systems☆

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ABSTRACT

A model of the dynamics of a mechanical system that includes bodies whose contact is accompanied by the appearance of small relative displacement and brittle failure of their surfaces with the resultant appearance of an interlayer of chipped off elements between the bodies is constructed. The variation of the potential energy of the system upon displacement of the bodies through interlayer elements is investigated. Cases in which the limit model obtained when the rate of the relative displacement of the bodies reaches a zero value differs from the classical non-holonomic model of the dynamics of the system are identified. Sufficient conditions, which enable evaluation of the error and the boundaries of the applicability of the limit model, are formulated. ©2017

1. Introduction. Model of the process

We will consider a mechanical system with holonomic stationary ideal constraints, which contains a dynamically connected pair of bodies (a kinematic pair[1]) with one relative degree of freedom. In the absence of a load along the normal to the direction of displacement of the bodies, they come into contact at the point O. After loading, the shape of the bodies remains unchanged except in the vicinity of the point O, where they undergo small strains. For the sake of simplicity, the mechanical characteristics of the strained parts of the bodies are assumed to be identical. It is assumed that the bodies are fairly rigid and have mismatched shapes[2] and that the dimensions of their contact areas are consequently small compared with the dimensions of each of the bodies and the radii of curvature of their surfaces near the point O. When the dynamics of such systems are analysed in theoretical mechanics, the small strains of the bodies in a kinematic pair are often neglected, and within the assumption that they are absolutely hard, a non-holonomic model with the constraint

$$v = 0 \tag{1.1}$$

where ν is the rate of relative displacement of the bodies, which is equal to the velocity of the contact point O, is considered.

The sufficient conditions for the realization of the constraint can be found using the approach proposed by Carathéodory for analysing systems with viscous friction and subsequently extended to systems of a general form under the action of some forces, which are determined by the properties of the mechanical system under consideration (Refs 3–6, etc., see the review in Ref. 6). Within the framework of this approach in problems of dynamics not associated with studying the vicinity of the contact area, the interaction of the bodies is assumed to be a point interaction, and the distributions of the tangential and normal forces are replaced by the resultant forces, which are assumed to be applied at the point O (the resultant torque over the contact area is equal to zero). The expressions for the resultant forces are assigned empirically or on the basis of heuristic arguments. As a rule, the weak deformability property of the bodies is taken into account by a model of the tangential component of the contact force (the friction force), which is continuously dependent on the variable ν. The system obtained in the limit as $\nu \to 0$ is called a system under constraint (1.1), and the conditions for the existence of the limit are the conditions for the realization of this constraint for the model of the friction force adopted.

The possibility of a limit transition to other non-classical models under constraint (1.1) was discussed by Kozlov[7,8] in the example of a mechanical system with viscous friction, i.e., $\beta\nu/\varepsilon$, and an anisotropic inertia tensor obtained by adding the Lagrange function of the term $\alpha\nu^2/(2\varepsilon)$ (the kinetic energy of the "virtual mass") to it. Here α and β are positive constants, the parameter ε ($0 < \varepsilon \ll 1$) characterizes the smallness of ν compared to the characteristic velocity of the centre of mass of the system or other components of it. The limit as $\varepsilon \to 0$,

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which ensures that the condition $\nu \to 0$ is met, was used to construct a mathematical model[7,8] of the motion of a system under constraint (1.1), whose form is defined by the values of the parameter $\mu = \alpha/\beta$. When $\mu = 0$, the limiting (or so-called "intermediate") model goes over to the classical non-holonomic model, and when $\mu \to \infty$, it goes over to the "vaconomic" model, which can be obtained formally, if the model under constraint (1.1) is constructed using the Hamilton–Ostrogradskii variational principle instead of the D'Alembert–Lagrange principle, used traditionally in the dynamics of non-holonomic systems.

A non-classical model under constraint (1.1), which takes into account the influence of brittle failure of the near-surface layers of the bodies in a kinematic pair on the dynamics of the mechanical system under consideration, will be constructed below.

Such a model will be constructed by studying the motion of a mechanical system on three scale levels of the geometrical, force and time parameters. On the first level, the dynamics of the system and its components, namely, the components that are "slow" compared to the processes occurring within and in the vicinity of the contact area of the bodies (the second and third scale levels), the motions of the centres of mass, the motions relative to the centres of mass and so on under the assigned forces and torques is analysed using the approaches of theoretical mechanics. Since the bodies in a kinematic pair have mismatched shapes, their interaction is assumed to be a point interaction; the model of the contact force at the point O and the model of the influence of failure of the bodies on the dynamical properties of the system are shaped after analysing the fast processes that develop on two successive scale levels. We will assume that as a result of local brittle failure of the near-surface layers of the bodies in a kinematic pair, fine particles are chipped off from them and are subsequently transformed into an interlayer between the bodies. On the second scale level (the level of faster motions) the contact interaction between the bodies is analysed: a model of their deformation in the vicinity of the contact area that is distant from the near-surface layers is constructed, and a model of failure and a model of the friction of the bodies that have lost their integrity, which takes into account that displacement of the bodies begins to occur on part of the area of contact with the interlayer by means of rolling through its elements, are constructed. On the third scale level (the level of the fastest motions) the possible structures of the arrangement of the interlayer elements along the contact area of the bodies in a kinematic pair, which has a direct influence on the conditions of friction of the bodies, are considered.

Since the rates and ranges of variation of the variables of the scale levels just enumerated differ significantly, during the discussions we will use the ideas of separating the motions (outer and inner expansions), within which the fast processes are assumed to be completed when the slow motions are investigated, while the values of the variables that are slow relative to them are assumed to remain unchanged when the fast processes are investigated.

We will now proceed to a brief description of the processes that occur on each of the scale levels and to a list of the main assumptions of the work.

1.1. *Second scale level. Model of the deformation and failure of the bodies. Variation of the potential energy of the system*

We introduce the right-handed orthogonal system of coordinates $Oxyz$ with the Ox axis oriented along the direction of displacement of the bodies and the Oz axis oriented along the direction of their loading. We will assume that within the contact area the bodies interact in accordance with Coulomb's friction law. We will restrict ourselves to the case in which there is no sliding of the particles as whole bodies (such will be the case, for example, when they roll), i.e., the contact area is divided into slip zones, where the displacements of the bodies relative to one another in the tangential direction (along the Ox axis) are non-zero, and cohesion zones, in which the displacements indicated are identical (see Ref. 2, Fig. 7.6). The coefficient of friction κ of the surfaces of the bodies is assumed to be fairly large, and then the cohesive zone will significantly exceed the slip zone[2] (in the case of rolling of the bodies, slipping will occur mainly in the small zone where they leave the contact area). Neglecting the slip zone, we will assume that the distributions of the tangential (τ) and normal (σ, along the Oz axis) stresses over the contact area satisfy the condition of cohesion, i.e., the tangential stresses do not exceed the limit values of the stresses of the Coulomb friction

$$|\tau| < \kappa\sigma \tag{1.2}$$

By virtue of inequality (1.2), the tangential displacements of points in the contact area of the bodies are identical; therefore, the velocity of these points, which is equal to the velocity ν of the point O, coincides with the difference between the velocities of the undeformed contours of the bodies, which demarcate the deformable neighbourhood of the point O. Then the tangential strains u_x of the bodies satisfy the equation

$$\dot{u}_x = \upsilon/\rho \tag{1.3}$$

where the dot denotes differentiation with respect to the time t, and ρ is the width of the contact area.

In accordance with the velocity addition theorem, the following equality is valid:

$$\upsilon = \upsilon^e + \upsilon^r \tag{1.4}$$

where ν^e is the translational component of the velocity of the point O together with system $Oxyz$ (the difference between the translational velocities of the non-deformable contours of the bodies), and ν^r is the component of the velocity of the point O relative to this system (the difference between the relative velocities of the non-deformable contours of the bodies). In accordance with inequality (1.2), ν^e and ν^r satisfy the condition

$$|\upsilon| \ll |\upsilon^e| \sim |\upsilon^r| \tag{1.5}$$

Assuming that the bodies are deformed significantly faster in the normal direction than in the tangential direction, we will consider these processes separately. We will assign the relation between the tangential and normal stresses and strains of the bodies in the vicinity of the contact area according to Ishlinskii's model (Ref. 9, Vol. 1), which is also called the "standard body model" (Ref. 9, Vol. 1, p. 13) or the "generalized linear medium model" (Ref. 10). This model of viscoelasticity, based on the linear relation between the stress, the strain and their derivatives with respect to time, enables us to describe the behaviour of a material which has not only elastic properties, but also relaxation (the ability to alter its stress state at a constant strain) and an aftereffect (the ability to alter its strain state at a constant stress).

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