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On stability in a case of oscillations of a pendulum with a mobile point mass☆

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ABSTRACT

Motion in a uniform gravitational field of a modified pendulum in the form of a thin, uniform rod, one end of which is attached by a hinge, is investigated. A point mass (for example, a washer mounted on the rod) can move without friction along the rod. From time to time, the point mass collides with the other end of the rod (if, for example, at this end of the rod a rigid plate of negligibly small mass is attached perpendicular to it). The collisions are assumed to be perfectly elastic. There exists such a motion of the pendulum in which the rod is at rest (it hangs) along the vertical passing through its suspension point, but the point mass moves along the rod, periodically bouncing up from its lower end to some height not exceeding the rod length. The nonlinear problem of the orbital stability of this periodic motion of the pendulum is investigated. In the space of two dimensionless parameters of the problem, stability and instability regions are found.

1. The equations of motion. Particular periodic solution

Let M be the mass of the rod, l be its length, and m be the mass of the point P carried by the rod. The motion takes place in a fixed vertical plane passing through the suspension point O of the rod. We assign the pendulum position by the two parameters; θ and ξ, where θ is the angle between the rod and the vertical and ξ is the distance from the point P to the end A of the rod (Fig. 1). A unilateral constraint holds

$$\xi \geq 0 \tag{1.1}$$

For the kinetic and potential energy of the pendulum, we have the expressions

$$T = \frac{1}{6}[Ml^2 + 3m(l-\xi)^2]\dot{\theta}^2 + \frac{1}{2}m\dot{\xi}^2$$

$$\Pi = \frac{1}{2}[Mgl + 2mg(l-\xi)](1-\cos\theta) + mg\xi$$

The dot above a symbol denotes differentiation with respect to time t, and g is the acceleration due to gravity. For $\xi > 0$ (constraint (1.1) relaxed), the pendulum motion is described by equations with the Lagrange function $L = T - \Pi$. For $\xi = 0$ (constraint (1.1) stressed); the character of the pendulum motion is a perfectly elastic collision; therefore[1]

$$\dot{\xi}^+ = -\dot{\xi}^- \tag{1.2}$$

Here and below, the minus and plus superscripts indicate values of the corresponding quantities immediately before and after a collision.

The equations of motion admit a particular solution, in correspondence with which the rod occupies a vertical position the entire time, and the point P moves up and down along the rod, periodically colliding with the lower end A of the rod. If h is the height of a bounce of

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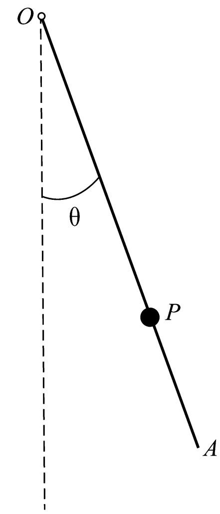

Fig. 1.

the point P above the point A ($0<h<1$), the above-mentioned particular solution corresponds to periodic motion of the pendulum with period $\tau_* = 2\sqrt{2h/g}$, equal to the time interval between successive collisions. For $0 \le t < \tau_*$, periodic motion is described by the equalities

$$\theta = 0, \qquad \xi = -\frac{g}{2}t^2 + \sqrt{2gh}\,t \tag{1.3}$$

As the initial time $t=0$, we take the time when $\xi=0$, and the velocity of the point P is directed upwards and is equal to $\sqrt{2gh}$. The main goal of this paper is to solve the problem of the orbital stability of periodic motion of the pendulum (1.3).

2. The Hamilton function

In what follows, to describe the pendulum motion we will use the Hamiltonian form of the equations of motion. If we introduce dimensionless coordinates and momenta x, y, *and* p_x, p_y, and transform to a new independent variable τ according to the formulae

$$\xi = lx, \quad \theta = y, \quad \frac{\partial L}{\partial \dot{\xi}} = \pi m l \sqrt{\frac{g}{2h}} p_x, \quad \frac{\partial L}{\partial \dot{\theta}} = \pi m l^2 \sqrt{\frac{g}{2h}} p_y, \quad \tau = \frac{2\pi}{\tau_*}t \tag{2.1}$$

then we obtain the following expression for the Hamilton function:

$$G = \Gamma + \frac{p_y^2}{2[\delta + (1-x)^2]} + \frac{\alpha}{\pi^2}[3\delta + 2(1-x)](1-\cos y) \tag{2.2}$$

where

$$\Gamma = \frac{1}{2}p_x^2 + \frac{2\alpha}{\pi^2}x \tag{2.3}$$

We have introduced the dimensionless parameters

$$\alpha = \frac{h}{l}\ (0 < \alpha < 1), \quad \delta = \frac{M}{3m} \tag{2.4}$$

For $x>0$, the pendulum motion is described by canonical differential equations with Hamilton function (2.2). For $x=0$ (at the moment of impact) we have

$$p_x^+ = -p_x^- \tag{2.5}$$

In the new variables x, y, p_x, p_y *and* τ, periodic motion of the pendulum (1.3) is written in the form

$$x = \alpha\varphi_2(\tau), \quad p_x = \frac{2\alpha}{\pi}\varphi_1(\tau), \quad y = 0, \quad p_y = 0 \tag{2.6}$$

where φ_1 and φ_2 are 2π-periodic functions of τ. For $0 \le \tau < 2\pi$, they are given by

$$\varphi_1(\tau) = 1 - \frac{\tau}{\pi}, \qquad \varphi_2(\tau) = \frac{\tau}{\pi}\left(2 - \frac{\tau}{\pi}\right) \tag{2.7}$$