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On periodic motions of a gyrostat satellite with a large inner (gyrostatic) angular momentum☆

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ABSTRACT

Rotational motion of an axially symmetric gyrostat satellite under the action of its gravitational torque in a circular orbit is considered. Periodic motions of the symmetry axis of the satellite relative to the orbital coordinate system are investigated. In absolute space, these motions appear as a slow precession about the normal to the orbital plane. Such motions are described by an autonomous system of fourth-order differential equations. The gyrostatic angular momentum is assumed to be large, which allows us to introduce a large parameter into the equations of motion. Solutions that are a rest in absolute space, with the symmetry axis forming a nonzero angle with the orbital plane, serve as the generating solutions. The period of the found solutions depends on this angle. Earlier, the limit case of such periodic motions when the symmetry axis of the satellite lies in the orbital plane in the generating solutions was investigated. The limit solutions describe small oscillations of the symmetry axis of the satellite in absolute space, and their period is equal to half the orbital period. To prove the existence of the new motions, we reduce the boundary value problem configuring the periodic solutions to a system of integral equations, which is solved by the method of successive approximations. This reduction is carried out according to the same scheme as in the degenerate case, but the necessary solutions of the integral equations are constructed differently. The result obtained explains the appearance of the limit solutions although the latter cannot be constructed within the framework of the considered general case.

1. Equations of motion and their properties

Let us consider an axially symmetric gyrostat satellite, whose centre of mass moves along an unknown circular orbit about a fixed attractive centre. The gyrostatic angular momentum of the satellite is constant and directed along its axis of material symmetry. To write out the equations of rotational motion of the satellite, we introduce two right-handed Cartesian coordinate systems.

The system $x_1x_2x_3$ is formed by the Rezal axes [1] of the satellite, and the x_1 axis is directed along its axis of material symmetry. The system $Y_1Y_2Y_3$ is the orbital coordinate system, and the Y_3 axis is directed along the radius vector of the centre of mass relative to the attractive centre, and the Y_2 axis is perpendicular to the orbital plane of the satellite and is directed along the kinetic moment vector of its orbital motion. The system $Y_1Y_2Y_3$ rotates about the Y_2 axis with constant angular velocity ω_0 equal to the average motion of the centre of mass of the satellite.

The orientation of the system $x_1x_2x_3$ relative to the system $Y_1Y_2Y_3$ is given by the angles α and β. The system $Y_1Y_2Y_3$ is rotated into the system $x_1x_2x_3$ by the two rotations (here we assume that these systems have a common origin): 1) through the angle $\alpha+\pi/2$ about the Y_2 axis, 2) through the angle β about the Y_3 axis obtained after the first rotation. The geometrical sense of these angles is the following: β is the angle between the x_1 axis and the orbital plane Y_1Y_3, $\beta>0$ if this axis is directed into the half-space $Y_2>0$; α is the angle between the $-Y_3$ axis and the projection of the x_1 axis onto the Y_1Y_3 plane, with the direction in which this angle is measured being in agreement with the direction of the Y_2 axis.

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Of the external torques applied to the satellite, we will only take the gravitational one into account. In this case, the theorem on the variation of the kinetic moment of rotational motion of the satellite is written as follows:

$$\frac{d\mathbf{K}}{dt} = 3\omega_0^2(I_1 - I_2)(\mathbf{n}\cdot\mathbf{E}_3)(\mathbf{E}_3\times\mathbf{n}) \tag{1.1}$$

Here $\mathbf{K}$ is the kinetic moment of the satellite in its motion relative to the centre of mass. I_1 and I_2 are the polar and equatorial principal central moments of inertia of the satellite, $\mathbf{n}$ is the unit vector of the x_1 axis and $\mathbf{E}_3$ is the unit axis of the Y_3 axis.

In the case $|\mathbf{K}| \approx \mathbf{K}\cdot\mathbf{n} = K_1$ we take $\mathbf{K} = K_1\mathbf{n}$. Substituting this approximate relation into Eq. (1.1), we obtain

$$\frac{d\mathbf{n}}{dt} + \frac{\mathbf{n}}{K_1}\frac{dK_1}{dt} = \kappa(\mathbf{n}\cdot\mathbf{E}_3)(\mathbf{E}_3\times\mathbf{n}),\quad \kappa = \frac{3\omega_0^2(I_1 - I_2)}{K_1}$$

The scalar product of the latter equation by $\mathbf{n}$ gives $dK_1/dt = 0$. Consequently, K_1 can be taken as a parameter. We write the equation

$$\frac{d\mathbf{n}}{dt} = \kappa(\mathbf{n}\cdot\mathbf{E}_3)(\mathbf{E}_3\times\mathbf{n}) \tag{1.2}$$

in the orbital coordinate system $Y_1Y_2Y_3$. In this case

$$\frac{d\mathbf{n}}{dt} = \frac{\tilde{d}\mathbf{n}}{dt} + \omega_0\mathbf{E}_2\times\mathbf{n},\quad \mathbf{n} = (-\sin\alpha\cos\beta, \sin\beta, -\cos\alpha\cos\beta)$$

$$\mathbf{E}_2 = (0,1,0),\quad \mathbf{E}_3 = (0,0,1)$$

The symbol $\tilde{d}/dt$ denotes the local derivative of the vector in the orbital system, and $\mathbf{E}_2$ is the unit vector of the Y_2 axis. Substituting the above expressions into Eq. (1.2), we obtain three equations in the derivatives $d\alpha/dt$ and $d\beta/dt$. These equations are consistent since Eq. (1.2) admits the integral relation $\mathbf{n}\cdot\mathbf{n} = 1$. We have

$$\frac{d\alpha}{dt} + \omega_0 = -\kappa\cos^2\alpha\sin\beta\,,\quad \frac{d\beta}{dt} = \kappa\sin\alpha\cos\alpha\cos\beta \tag{1.3}$$

Equations (1.3) coincide with the so-called evolutionary equations obtained and investigated by V. V. Beletskii [2] (in place of the angle β he used the angle $\pi/2 - \beta$). More accurately, Eqs (1.3) are a particular case of them, corresponding to rapid rotation of an axially symmetric satellite about its symmetry axis. Beletskii considered the more general problem, when rapid rotation of such a satellite is close to regular Euler precession. However, the transformation to the more general case only alters the form of the constant factor κ in an unfundamental way.

Equations (1.3) admit a first integral:

$$\sin\beta - \frac{\kappa}{2\omega_0}\cos^2\alpha\cos^2\beta = \mathrm{const}$$

and can be integrated. Beletskii found the exact general solution of Eqs (1.3) in elliptic Jacobi functions, and also, applying the method of averaging, obtained approximate formulae for the solution for $|\kappa| \ll \omega_0$.

In this case, α is a fast variable, and β is a slow variable; averaging of Eqs (1.3) over α gives

$$\frac{d\alpha}{dt} = -\omega_0 - \frac{\kappa}{2}\sin\beta,\quad \frac{d\beta}{dt} = 0$$

In the general solution of the averaged equations, $\beta = \mathrm{const}$ and α is a linear function of time. For $|\kappa| \ll \omega_0$ the exact solutions of Eqs (1.3), lying outside a small neighbourhood of the poles $\beta = \pm\pi/2$ of the sphere $\mathbf{n}\cdot\mathbf{n} = 1$, are periodic. For each solution, a number $T > 0$ (period), exists such that in this solution

$$\alpha(t+T) \equiv \alpha(t) - \pi,\quad \beta(t+T) \equiv \beta(t)$$

The periodicity condition for α is written with allowance for the circumstance that this variable enters into Eqs (1.3) periodically with period π. The period of motion of the x_1 axis, described by such a solution, with respect to the orbital coordinate system is equal to $2T$.

Below, we assume that $|\mathbf{K}| \approx K_1$ and that $|\kappa| \ll \omega_0$. We prove the existence of periodic motions of the unit vector $\mathbf{n}$ with respect to the orbital coordinate system that are close to the solutions of Eqs (1.3). In order to describe the problem in greater detail, we write relation (1.1) in the system $x_1x_2x_3$. In this case,

$$\frac{d\mathbf{K}}{dt} = \frac{\tilde{d}\mathbf{K}}{dt} + \boldsymbol{\omega}\times\mathbf{K},\quad \mathbf{K} = (K_1, I_2\omega_2, I_2\omega_3),$$

$$\boldsymbol{\omega} = (\omega_1,\omega_2,\omega_3),\quad \mathbf{n} = (1,0,0),\quad \mathbf{E}_3 = (-\cos\alpha\cos\beta, \cos\alpha\sin\beta, -\sin\alpha)$$

$$\omega_1 = \left(\frac{d\alpha}{dt} + \omega_0\right)\sin\beta,\quad \omega_2 = \left(\frac{d\alpha}{dt} + \omega_0\right)\cos\beta,\quad \omega_3 = \frac{d\beta}{dt}$$