Journal of Applied Mathematics and Mechanics xxx (2018) xxx–xxx

Contents lists available at ScienceDirect

Journal of Applied Mathematics and Mechanics

journal homepage: www.elsevier.com/locate/jappmathmech

# Spring analogy of non-linear oscillations of a bubble in a liquid at resonance☆

V.V. Vanovskii*, A.G. Petrov

*Institute for Problems in Mechanics, Russian Academy of Sciences, Moscow, Russia*

ARTICLE INFO

*Article history:*
Received 20 October 2016
Available online xxx

ABSTRACT

Two non-linear oscillatory systems are considered. The first is a point mass on a spring with vertical vibration of the suspension point with a frequency that coincides with the frequency of free vertical oscillations and is two times greater than the frequency of free horizontal oscillations. The friction force in the spring is taken into account. For an initial deviation of the point mass from the vertical, after a long enough time the energy of the vertical oscillations is almost completely transferred into the energy of horizontal oscillations. Using an averaging method, an asymptotic solution is constructed, describing the transient process setting up a periodic solution. Comparison of the analytical solution with the numerical one demonstrates its high accuracy. The second system is an axisymmetrical bubble in a liquid under the variable pressure. An analogy between this system and the previous one is established. Vibration of the suspension point of a spring pendulum corresponds to variable liquid pressure, and the vertical and horizontal oscillation modes of the swinging spring correspond to the radial and deformational oscillation modes of the bubble, and the ratio of the frequencies of these modes is also taken to be equal to two. The friction force in the spring corresponds to energy dissipation under radial oscillations of the bubble. In our calculations of energy dissipation, we take into account the liquid viscosity, thermal dissipation, and acoustic radiation due to liquid compressibility. During transfer of the energy of the radial oscillations, the amplitude of the resonant deformational mode of the bubble oscillations grows anomalously, which makes it possible for the bubble to break up with small energy dissipation under the action of a time-varying external pressure field.



## 1. Introduction

The problem of free planar oscillations of a swinging spring at the 2 : 1 resonance was introduced by Vitt and Gorelik[1] in a description of an experiment in which periodic transfer of the energy of the vertical oscillations of a pendulum into horizontal oscillations was observed.

It has been remarked (Ref. 2, p. 106) that this experiment was proposed by Mandel'shtam to illustrate some peculiarities of the vibrations of the $CO_2$ molecule, which Fermi had discovered[3] while analysing the Raman spectra of gaseous $CO_2$. In the $CO_2$ molecule the frequency of the longitudinal mode practically coincides with twice the frequency of the transverse mode, which leads to a modulation regime of energy exchange between vibration modes of the molecule. This leads to the appearance of additional bands in the Raman spectra. A spring pendulum with two oscillatory degrees of freedom is the simplest classical analogue of such a system, and Mandel'shtam, Vitt and Gorelik also turned their attention to it.

Since that time, the problem of the free oscillations of a swinging spring has attracted the attention of many investigators. There is a review[4] of the results of a qualitative analysis of this problem. In particular, it has been shown that oscillations strictly along the vertical are unstable. For a small deviation along the horizontal, there is a gradual transition of the vertical oscillations into horizontal oscillations, and the main asymptotic limit of the transfer period has been calculated.

An asymptotic solution of the planar problem at the 2 : 1 resonance has been constructed, describing the periodic process of energy transfer in terms of elementary functions,[5,6] and an asymptotic solution in the case of forced planar oscillations of a swinging spring with

☆ *Prikl. Mat. Mekh.* Vol. 81, No.4, pp. 445–461, 2017.

* Corresponding author.
*E-mail address:* vladimir.vanovsky@gmail.com (V.V. Vanovskii).

https://doi.org/10.1016/j.jappmathmech.2017.12.008
0021-8928/

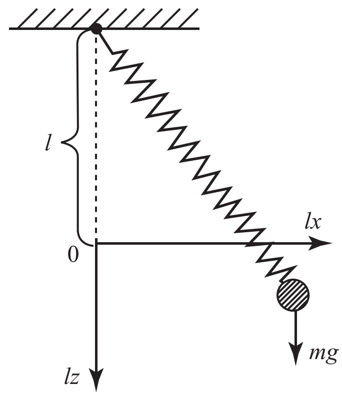


**Fig. 1.**

dissipation in the spring taken into account has also been constructed, where the vibration frequency coincides with the frequency of the vertical natural oscillations and is twice the natural frequency of oscillations along the horizontal.[7]

In two previous papers,[8,9] we established an analogy between the natural oscillations of a spring and those of a bubble in a liquid under resonance conditions. We showed in these papers that as energy is transferred, the amplitude of the deformational mode grows, and this can be one of the mechanisms of bubble breakup. A substantial shortcoming of these works is the absence of dissipation in the models proposed. Energy losses attendant to radial oscillations of a bubble lead to the result that the free radial oscillations of the bubble die out before energy transfer to the deformational mode begins.

The calculations presented below fill this gap. What is new in the formulation that follows consists in an investigation of forced non-linear oscillations of a bubble in a liquid, taking dissipation into account, under the external pressure varying with the frequency of the free radial oscillations of the bubble. As a result of energy being pumped in by the external variable pressure in parallel with energy dissipation, periodic non-decaying oscillations of the bubble are set up. The oscillations are investigated under the condition of a 2 : 2 :1 resonance between the frequencies of the external pressure field and the radial and deformational oscillation modes of the bubble. The main idea, with the help of which we realised this study, was to apply methods developed earlier to solve the problem of non-linear forced oscillations of a swinging spring[7] and to establish an analogy between these two problems.

It was shown that the effect of resonant energy transfer can lead to a quite large amplitude of the deformational oscillations of a bubble for a relatively small amplitude of the exciting pressure wave, which can bring about bubble breakup, as was observed in experiments.[10] This effect can have important technological and medical applications, for example, in breaking through the hemato-encephalic barrier,[11] and also serve as one of the mechanisms for revealing the sub-harmonics in the spectrum of an irradiated bubble.[12]

## 2. Statement of the problem on forced oscillations of a swinging spring.

In a Cartesian coordinate system with its $x$ and $z$ axes directed in the horizontal and vertical directions, respectively, we consider a pendulum with two degrees of freedom: a heavy point swinging while suspended from a weightless spring (Fig. 1). The suspension point vibrates in the vertical with acceleration $w_0 = -a\Omega'^2 \cos \Omega' t'$, and the rest position of the load, the point $O$, is located at the origin of the coordinate system $x = 0$, $z = 0$. We introduce the following notation: $c$ is the stiffness of the spring, $l_0$ is the length of the unloaded spring, $l$ is its length for the rest position of the load, $m$ is the mass of the load, $lx$ and $lz$ are the coordinates of the load, $lR$ is the length of the spring, and $R = \sqrt{x^2 + (1+z)^2}$. The tension of the spring is proportional to its elongation $T = c(lR - l_0)/l_0$.

It is convenient to investigate non-linear oscillations using the Hamiltonian form of the equations. The potential $E_p$ and kinetic $E_c$ energy of the system in the non-inertial coordinate system bound to the suspension point have the form (the dot above a symbol denotes differentiation with respect to dimensionless time $t$, dimensional time is $t'$)

$$E_p = \frac{c\left(lR - l_0\right)^2}{2l_0} - mg'lz, \quad g' = g - w_0$$

$$E_c = \frac{mgl}{2}\left(\dot{x}^2 + \dot{z}^2\right), \quad t = \sqrt{\frac{g}{l}}t'$$

In this system there are three frequencies: the Huygens frequency of the horizontal oscillations $\omega_x$ and the vertical oscillations frequency $\omega_z$:

$$\omega_x = \sqrt{\frac{g}{l}}, \quad \omega_z = \sqrt{\frac{g}{l}K(\lambda+1)}; \quad K = \sqrt{\frac{c}{mg}}, \quad \lambda = \frac{l}{l_0} - 1$$

and the prescribed vibration frequency of the suspension point $\Omega'$. It follows from the equilibrium conditions that $K\lambda = 1$, and from the resonance conditions $\Omega' = \omega_c = 2\omega_x$ we obtain $K = 3$ and $\lambda = 1/3$.

Please cite this article in press as: Vanovskii VV, Petrov AG. Spring analogy of non-linear oscillations of a bubble in a liquid at resonance. *J Appl Math Mech* (2018), https://doi.org/10.1016/j.jappmathmech.2017.12.008