G Model
JAMM-2400; No. of Pages 6
ARTICLE IN PRESS
Journal of Applied Mathematics and Mechanics xxx (2017) xxx–xxx
Contents lists available at ScienceDirect
Journal of Applied Mathematics and Mechanics
journal homepage: www.elsevier.com/locate/jappmathmech

The evolution of the motions of a rigid body close to the Lagrange case under the action of an unsteady torque☆

L.D. Akulenko[a,*], YA.S. Zinkevich[b], T.A. Kozachenko[c], D.D. Leshchenko[d]

[a] Institute for Problems in Mechanics, Russian Academy of Sciences, Moscow, Russia
[b] Moscow Institute of Physics and Technology, Dolgoprudny, Russia
[c] Bauman Moscow State Technical University, Moscow, Russia
[d] Odessa State Academy of Civil Engineering and Architecture, Odessa, Ukraine

ARTICLE INFO

Article history:
Received 7 September 2016
Available online xxx

ABSTRACT

Perturbed rotational motions of a rigid body, close to the Lagrange case, under the action of a torque that is slowly varying in time are investigated. Conditions for the possibility of averaging the equations of motion with respect to the nutation phase angle are presented and an averaged system of equations is obtained. An example, corresponding to the motion of a body in a medium with linear dissipation, is considered.

The problem of the evolution of the rotations of a rigid body about a fixed point continues to attract attention related to problems of the entry of aircraft into the atmosphere, cosmonautics, gyroscopes and the dynamics of a rotating missile. Here, in many cases, a motion in the Lagrange case can be considered as the generating motion of the body that takes account of the main torques acting on it. In this case, the body has a fixed point and is in a gravitational force field, and, moreover, the centre of mass of the body and the fixed point lie on the axis of dynamic symmetry of the body. A restoring torque similar to the gravitational torque, is created by the aerodynamic forces acting on the body in a gas stream. Motions close to the Lagrange case have therefore been investigated in a number of papers on aerodynamics where the restoring torque and different perturbing torques were taken into account.

We mention the analysis of the motion about the centre of mass of aircraft entering the atmosphere at a high velocity.[1,2] The motion of a rotating rigid body in the atmosphere under the action of a sinusoidal or biharmonic time-dependent restoring torque and small perturbing torques has been investigated.[3] A procedure for averaging over an Euler–Poinsot motion for a satellite with an arbitrary triaxial ellipsoid of inertia was constructed for the first time.[4] The perturbed motions of a rigid body, close to the Lagrange case, have been considered in a number of papers such as Refs 5–14, for example. A review of the results obtained up to 1998 on the problem of the evolution of the rotations of a rigid body, close to the Lagrange case, is available.[10] An averaging procedure for the slow variables in the first approximation of the perturbed motion of a rigid body, close to the Lagrange case, has been described and the perturbed rotational motions of a rigid body, close to a regular precession in the Lagrange case, have been studied for different orders of smallness of the projections of the angular momentum vector.[5,6,14]

The motion of a heavy symmetric rigid body with a fixed point under the action of friction forces due the surrounding dissipative medium has been considered.[7] The asymptotic behaviour of the motions of a Lagrange gyroscope, close to regular precessions, under the action of a small perturbing torque has been investigated.[8,9,12] The motion of a Lagrange gyroscope with a vibrating suspension has been studied [11] and the effect of fast periodic and conditionally periodic vibrations of the point of suspension on the existence and stability of the steady rotations of a Lagrange gyroscope about the vertical and its regular precessions [13] have been studied.

☆ *Prikl. Mat. Mekh.*, Vo. 81, No. 2, pp. 115–122, 2017.

* Corresponding author.
E-mail addresses: kumak@ipmnet.ru (L.D. Akulenko), yaninaz@mail.ru (YA.S. Zinkevich), kushpil.ru@rambler.ru (T.A. Kozachenko), leshchenko_d@ukr.net (D.D. Leshchenko).

http://dx.doi.org/10.1016/j.jappmathmech.2017.08.001
0021-8928/

1. Statement of the problem and unperturbed motion

The motion of a dynamically symmetric rigid body about a fixed point O in the case of perturbations of an arbitrary physical nature is considered. The equations of motion (the dynamic and kinematic Euler equations) have the form

$$\begin{aligned}
&A\dot{p} + (C - A)\,qr = \mu \sin\theta \cos\varphi + \varepsilon M_1 \\
&A\dot{q} + (A - C)\,pr = -\mu \sin\theta \sin\varphi + \varepsilon M_2 \\
&C\dot{r} = \varepsilon M_3; \quad M_i = M_i(p, q, r, \psi, \theta, \varphi, \tau), \quad i = 1, 2, 3 \\
&\dot{\psi} = (p \sin\varphi + q \cos\varphi)\operatorname{cosec}\theta, \quad \tau = \varepsilon t \\
&\dot{\theta} = p\cos\varphi - q\sin\varphi, \quad \dot{\varphi} = r - (p\sin\varphi + q\cos\varphi)\operatorname{ctg}\theta
\end{aligned} \tag{1.1}$$

Dynamic equations (1.1) are written in projections onto the principal axes of inertia of the body passing through the point O. Here, p, q, and r are the projections of the angular velocity vector onto these axes, M_i ($i = 1, 2, 3$) are the projections of the perturbing moment vector onto the same axes, and they depend on the slow time $\tau = \varepsilon t$, where $\varepsilon \ll 1$ is a small parameter characterizing the magnitude of the perturbations and t is the time), ψ, θ and φ are Euler angles, and A is the equatorial and C is the axial moment's of inertia about the point O $A \neq C$. It is assumed that a restoring torque acts on the body, the maximum magnitude of which is equal to μ and is created by a force, constant in magnitude and direction, that is applied to a certain fixed point of the axis of dynamic symmetry. In the case of a heavy top, we have $\mu = mgl$, where m is the body mass, g is the acceleration due to gravity and l is the distance from the fixed point O to the centre of gravity of the body.

We set the problem of investigating the asymptotic behaviour of the solutions of system (1.1) for a small ε; the analysis will be carried out by the method of averaging[15] on a time interval of the order of ε^{-1}.

In the case of the unperturbed motion, the quantities[14,16]

$$\begin{aligned}
&G_z = A\sin\theta\left(p\sin\varphi + q\cos\varphi\right) + Cr\cos\theta = c_1 \\
&H = \frac{1}{2}[A(p^2 + q^2) + Cr^2] + \mu\cos\theta = c_2, \quad r = c_3
\end{aligned} \tag{1.2}$$

are the first integrals of the equations for system (1.1) when $\varepsilon = 0$. Here, G_z is the projection of the angular momentum vector onto the vertical Oz, H is the total energy of the body, r is the projection of the angular velocity vector onto the axis of dynamic symmetry, c_i ($i = 1, 2, 3$) are arbitrary constants and $c_2 \geq -\mu$.

In the general case, the expression for the angle of nutation θ in the unperturbed motion is known as functions of the time t, the integrals of the motion (1.2) and the arbitrary phase constant β:[14,16]

$$\begin{aligned}
&u = \cos\theta = u_1 + (u_2 - u_1)\operatorname{sn}^2(\alpha t + \beta), \quad \operatorname{sn}(\alpha t + \beta) = \sin\operatorname{am}(\alpha t + \beta, k) \\
&\alpha = \left[\mu(u_3 - u_1)/(2A)\right]^{1/2}, \quad k^2 = (u_2 - u_1)(u_3 - u_1)^{-1}, \quad 0 \leq k^2 \leq 1
\end{aligned} \tag{1.3}$$

Here, u is the periodic function $\alpha t + \beta$ with a period $K(k)/\alpha$, sn and am are an elliptic sine and amplitude,[17] k is the modulus of the elliptic functions, and the real roots of the cubic polynomial

$$Q(u) = A^{-2}[(2H - Cr^2 - 2\mu u)(1 - u^2)A - (G_z - Cru)^2] \tag{1.4}$$

are denoted by u_1, u_2 and u_3.

The relations between its roots and first integrals (1.2) are written in the following way:

$$\begin{aligned}
&u_1 + u_2 + u_3 = \frac{H}{\mu} - \frac{Cr^2}{2\mu} + \frac{C^2r^2}{2A\mu}, \quad u_1u_2 + u_1u_3 + u_2u_3 = \frac{G_zCr}{A\mu} - 1 \\
&u_1u_2u_3 = -\frac{H}{\mu} + \frac{Cr^2}{2\mu} + \frac{G_z^2}{2A\mu} \\
&-1 \leq u_1 \leq u_2 \leq 1 \leq u_3 < +\infty
\end{aligned} \tag{1.5}$$

Formulae (1.2), (1.3) and (1.5) describe the solution of system (1.1) when $\varepsilon = 0$ in the Lagrange case.

2. Averaging procedure

An averaging procedure developed earlier[5,14] is used later for averaging system (1.1) in the case of perturbations depending on the slow time τ and allowing averaging over the phase of the nutation angle θ along the trajectories of change $\theta(t)$. We separate the fast and slow variables, and, here, first integrals (1.2) for the perturbed motion (1.1) are the slow variables. The fast variables are the angles of proper rotation φ, nutation 2 and precession ψ.

Using a number of transformations, we reduce the first three equations of (1.1) to the form[5,14]

$$\begin{aligned}
&\dot{G}_z = \varepsilon\left[(M_1\sin\varphi + M_2\cos\varphi)\sin\theta + M_3\cos\theta\right] \\
&\dot{H} = \varepsilon\left(M_1p + M_2q + M_3r\right) \\
&\dot{r} = \varepsilon C^{-1}M_3; \quad M_i = M_i\left(p, q, r, \psi, \theta, \varphi, \tau\right), \quad i = 1, 2, 3
\end{aligned} \tag{2.1}$$