G Model
JAMM-2393; No. of Pages 8

ARTICLE IN PRESS

Journal of Applied Mathematics and Mechanics xxx (2017) xxx–xxx

Contents lists available at ScienceDirect

Journal of Applied Mathematics and Mechanics

journal homepage: www.elsevier.com/locate/jappmathmech

Method for the analysis of long water waves taking into account reflection from a gently sloping beach☆

D.A. Lozhnikov*, V.E. Nazaikinskii

Institute for Problems in Mechanics of the Russian Academy of Sciences, Moscow, Moscow Institute of Physics and Technology (State University), Dolgoprudny, Russia

ARTICLE INFO

Article history:
Received 16 April 2016
Available online xxx

ABSTRACT

Algorithms and their software implementation are described that allow calculating the front propagation for a long wave (for example, a tsunami wave) described in the approximation of the linearized shallow water equations with reflection from a shallow beach taken into account. The algorithms are based on a construction of asymptotic solutions of the Cauchy problem for hyperbolic equations with degeneracy on the boundary earlier proposed with the participation of one of the authors. The results of a numerical experiment are presented.

1. Introduction

In the absence of vortex motions of the liquid, the propagation of long waves with the variable velocity $c(x)=\sqrt{gD(x)}$ in a basin of variable depth $D(x)$ $(x=(x_1,x_2))$ is described in the approximation of linearized shallow water equations by the wave equation

$$\eta_{tt}-\langle\nabla, c^2(x)\nabla\rangle\eta = 0 \tag{1.1}$$

for a scalar function $\eta=\eta(x,t)$ specifying the free surface elevation at time t at a point x (e.g., see Refs 1–3); here g is the acceleration due to gravity. Equation (1.1) is considered in a bounded domain $\Omega\subset\mathbb{R}^2$ (the basin). We assume that the boundary $\partial\Omega$ of Ω (the shoreline) is a smooth curve and that the depth function $D(x)$ is smooth in the closure $\bar{\Omega}$ of Ω, vanishes identically on $\partial\Omega$, and is strictly positive in the interior of Ω. Thus, the beach is gently sloping, which means that bottom everywhere approaches the shoreline at a non-zero angle to the vertical. We also assume that the condition $\nabla D(x)\neq 0$ is satisfied on $\partial\Omega$, which means that the tangent plane to the bottom is not horizontal at any point of the shoreline. We pose the Cauchy problem for Eq. (1.1) with the initial conditions

$$t = 0,\quad x\in\Omega:\ \eta = \eta^0\left(\frac{x-x^0}{\mu}\right),\quad \eta_t = 0 \tag{1.2}$$

where $\eta^0(z)$ is a smooth function sufficiently rapidly decaying together with its derivatives as $|z|\to\infty$, x^0 is an interior point of Ω, and $\mu\to 0$ is a small parameter. Conditions (1.2) correspond to the *piston model*,[1–3] in which a tsunami wave is generated by a source caused by an instantaneous uplift or subsidence of the ocean bottom. The source shape, position, and horizontal dimensions are determined by the function $\eta_0(z)$, the point x^0, and the parameter μ. Thus, the source is concentrated in a neighbourhood of x^0 for small μ; the case of a source located on the shoreline $\left(x^0\in\partial\Omega\right)$ is excluded.

Since the spatial part $\langle\nabla, c^2(x)\nabla\rangle$ of the wave operator in Eq. (1.1) degenerates on the boundary of the domain, it turns out that no classical boundary conditions are needed,[4] and the Cauchy problem is well posed in the class of functions with finite energy integral

$$J^2(t) = \frac{1}{2}\left(\|\eta_t\|^2_{L^2(\Omega)}+\|c(x)\nabla\eta\|^2_{L^2(\Omega)}\right)$$

☆ *Prikl. Mat. Mekh.*, Vol. 81, No. 1, pp. 33–44, 2017.

* Corresponding author.
E-mail addresses: lozhnikov_d@list.ru (D.A. Lozhnikov), nazaikinskii@yandex.ru (V.E. Nazaikinskii).

http://dx.doi.org/10.1016/j.jappmathmech.2017.07.003
0021-8928/

Please cite this article in press as: Lozhnikov DA, Nazaikinskii VE. Method for the analysis of long water waves taking into account reflection from a gently sloping beach. *J Appl Math Mech* (2017), http://dx.doi.org/10.1016/j.jappmathmech.2017.07.003

There are quite a few papers dealing with the propagation of long waves and their run-up on a gently sloping beach (in the linearized as well as in the full non-linear statement). These papers give both exact solutions for model (one-dimensional) problems and approximate (numerical) solutions in more general situations and present an analysis of these solutions (e.g., see Refs 5–10 and the literature cited therein). Solving the Cauchy problem with general initial conditions for Eq. (1.1) in which the velocity $c(x)$ corresponds to the real bathymetry of some part of the ocean requires vast hardware resources and/or much computer time. However, the situation for problem (1.1), (1.2) is fundamentally different: the presence of the small parameter μ in the initial conditions (1.2) permits one to apply asymptotic methods based on the theory of Maslov's canonical operator[11,12] to this problem and obtain an analytical-numerical method that combines simple closed-form analytic expressions with the numerical solution of ordinary differential equations alone. Thus, the computational resources needed to solve the problem are reduced dramatically. These constructions implement Maslov's general idea[13] for the development of asymptotic-numerical algorithms. Note that tsunami wave propagation for the case in which the source is given by the Dirac delta function was studied by the ray method (e.g., see Refs 14 and 15) but only under the assumption that there are no caustics.

The standard version of Maslov's canonical operator is intended for the construction of rapidly oscillating solutions. Its modification suggested[16–18] for the case of localized solutions, which is based on a representation of localized initial data via a parametric integral of the canonical operator on a Lagrangian manifold of a special form, cannot be applied to Eq. (1.1) directly because of the degeneracy for $x \in \partial\Omega$. The velocity $c(x)$ vanishes on $\partial\Omega$, the entire boundary is a special kind of caustic, and it was shown[19] (developing the ideas expressed in the one-dimensional case[20]) that the phase space Φ of the classical Hamiltonian system corresponding to this equation does not coincide with the cotangent bundle $T^*\Omega$ of the domain Ω; instead, it is obtained from $T^*\Omega$ by the addition of some set Φ_∞ of "ideal" points over the boundary $\partial\Omega$.

A theory of asymptotic solutions of the Cauchy problem (1.1), (1.2) was constructed,[21–24] and for an initial condition of a special form,[25] simple integration-free analytic formulae for the boundary values of the solution of problem (1.1), (1.2) at the points that are not "strong" singular points were presented.[26] The closed-form formulae provided by this theory express the solution of problem (1.1), (1.2) via solutions of a system of ordinary differential equations, namely, the Hamiltonian system on the phase space $\Phi = T^*\Omega \sqcup \Phi_\infty$ with Hamiltonian

$$H(x,p) = c(x)\sqrt{p_1^2 + p_2^2} \equiv c(x)|p| \tag{1.3}$$

(The Hamiltonian has this form on $T^*\Omega$ and extends by continuity to the subset $\Phi_\infty = \Phi \backslash T^*\Omega$.) The solutions of this system with the initial conditions

$$t = 0: x = x_0, \quad p = p_0; \quad |p_0| = 1 \tag{1.4}$$

take into account reflections from the beach and determine the wave front propagation; the solution of the original problem with localized initial conditions is concentrated near the wave front. Clearly, one can only solve the Hamiltonian system numerically for problems with real bathymetry. Thus, the solution of the Hamiltonian system proves to be the most computationally laborious part when practically implementing the procedures developed in the cited papers. In turn, computing the trajectories of the Hamiltonian system in a neighbourhood of the points where these trajectories meet the set Φ_∞ (i.e., essentially, computing the reflection of the trajectories from the beach) is the most complicated stage of the whole numerical procedure. Note that the reflection of the trajectories from a gently sloping beach is very different from the reflection described by the standard laws of geometric optics (which would occur if the beach were a vertical wall), and it is essentially for its adequate description that the phase space Φ is introduced.

In what follows, we construct algorithms and software that permit computing the trajectories of the Hamiltonian system and the front of the wave described by problem (1.1), (1.2) at various instants of time. The software implementation of the analytical formulae representing the solution itself based on this information is not discussed.

The geometric construction[19] of the space Φ and the Hamiltonian system on it are presented in Section 2. Section 3 contains a description of the algorithm, and the results of a numerical experiment based on the software implementation of this algorithm are given in Section 4.

2. Geometry of the Hamiltonian system corresponding to the degenerate wave equation

2.1. Asymptotic solution and wave front propagation

Since the problem involves a small parameter μ, it is natural to solve it by asymptotic methods, i.e., construct the asymptotics of the solution as $\mu \to 0$.

It is well known (e.g., see Refs 11 and 12) that (at least where is no degeneracy) the wave propagation is determined by the trajectories of the Hamiltonian system with Hamiltonian (1.3) corresponding to Eq. (1.1). In particular, the wave front at each instant of time is the curve formed by the projections onto the space $\mathbb{R}^2_x$ of the points corresponding to this instant on the trajectories of the Hamiltonian system

$$\dot{p} = -H_x(x,p) \equiv -|p|\nabla c(x), \quad \dot{x} = H_p(x,p) \equiv \frac{p}{|p|}c(x) \tag{2.1}$$

with the initial conditions

$$t = 0: x = x^0, \quad p = \begin{pmatrix} \cos\psi \\ \sin\psi \end{pmatrix}, \quad \psi \in [0, 2\pi) \tag{2.2}$$

i.e., the wave is being sent from the point x^0 at all possible angles Ψ.

The solution of problem (1.1), (1.2) is concentrated near the wave front for each t. In particular, it is localized for $t = 0$ in a neighbourhood of the point x^0 corresponding to the position of the source. The front at each subsequent instant of time is a closed curve, which is at first smooth and nearly circular and can later manifest turning (focal) points and points of self-intersection.