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Collision of a body with an elastic membrane☆

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ABSTRACT

Problems on oscillations of an elastic weightless membrane stretched over a rigid circular ring, induced by normal and tangential incidence on it of a body in the form of a disc of radius small compared to the ring radius, are considered. According to existing theory, the large two-dimensional deformations that arise are described by nonlinear partial differential equations, but the main parameters of motion of the membrane can be estimated using simpler models proposed here.

1. Central collision of a body with an elastic weightless membrane

We consider motion of a body of unit mass in the form of a disc of radius $\varepsilon \ll 1$ colliding with an elastic weightless membrane stretched across a rigid ring of unit radius (Fig. 1). As the unit of time we take the ratio of the ring radius to the body velocity at the moment of contact with the membrane. At the initial time, the disc and the membrane are found in the same plane, and their centres coincide. The initial velocity of the disc υ (at the time of collision with the membrane) is directed along the normal to the plane of the membrane (Fig. 1, Case *a*).

We choose a fixed coordinate system $Oxyz$ with origin at the centre of the ring with the z axis directed along the normal to it, and z_1 is the z coordinate of the centre of the disc ($z_1 \geq 0$). By virtue of symmetry (only the action of elasticity forces is taken into account), the disc centre will move along the z axis, stretching the membrane. The aim of the present work is to describe the motion of the disc under the action of elastic forces.

In what follows we consider membranes, the potential energy of an element of which upon extension of the membrane is proportional to its area $ds: dV = \mu\, dS$ (μ is the coefficient of surface tension or elasticity). In order to determine the elasticity forces, we must find the shape of the membrane being extended. Since the membrane is assumed to be weightless, the shape of its surface between the body and the support ring has the form of a catenoid[1] (Fig. 1, Case *a*). The meridian of the catenoid, lying in the right half-plane xz ($x > 0$) is a catenary[1]

$$x = C_1 \operatorname{ch} \frac{z - C_2}{C_1} \tag{1.1}$$

The constants C_1 and C_2 are determined from two prescribed points on the catenary: the point (1,0) on the fixed ring over which the membrane is stretched, and the point (ε, z_1) corresponding to the position of the disc, i.e.,

$$C_1 \operatorname{ch} \frac{C_2}{C_1} = 1, \quad C_1 \operatorname{ch} \frac{z_1 - C_2}{C_1} = \varepsilon, \quad 0 < C_1 < \varepsilon \tag{1.2}$$

Conditions (1.2) give two representations for the z coordinate of the disc centre z_1 in terms of the constant C_1:

$$z_1^{\pm} = C_1 \ln\left[\frac{\varepsilon}{C_1^2}\left(1 \pm \sqrt{1 - \frac{C_1^2}{\varepsilon^2}}\right)(1 + \sqrt{1 - C_1^2})\right] \tag{1.3}$$

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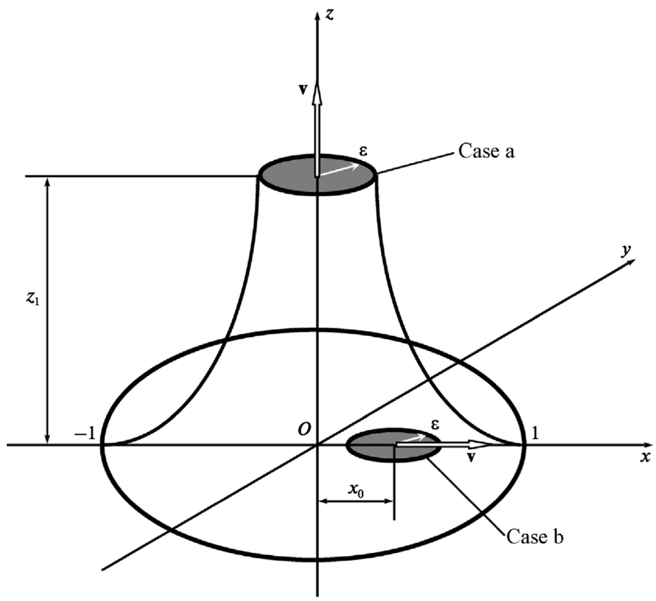

Fig. 1.

where

$$C_2 = C_1 \ln(C_1^{-1} + \sqrt{C_1^{-2} - 1}) \tag{1.4}$$

Leaving details aside, we can indicate the values $C_1 = C_1^+$ and $C_1 = C_1^-$ at which the derivatives with respect to the constant C_1 of the z coordinate of the disc centre z_1 from expressions (1.3) remain positive (the plus and minus superscripts correspond to those in expressions (1.3))

$$C_1^+ = \varepsilon\left\{1 + 2(1+\sqrt{1-\varepsilon^2})^{-2}\ln\left(\frac{1+\sqrt{1-\varepsilon^2}}{\varepsilon} - 2\right)\right\}^{1/2}, \quad C_1^- = \sqrt{2\varepsilon}\exp\left(-\frac{1}{4} - \frac{1}{\sqrt{1-\varepsilon^2}}\right)$$

(For example, for $\varepsilon = 0.1$ we have $C_1^+ \approx 0.086$ and $C_1^- \approx 0.12$.)

It can be shown that at least for $\varepsilon < 0.1$ the value of $C_1^- > \varepsilon$. This guarantees monotonicity of the function $z_1^\pm$ in the interval $0 < C_1 < C_1^- < \varepsilon$. Numerical calculations show that monotonicity of dependences (1.3) is preserved for values of $\varepsilon < 0.1$. In this case, a disc of radius ε, as it stretches the membrane, is found below the minimum cross section of the catenoid as long as $C_1 < \varepsilon$. (For choice of the plus sign in expression (1.3), on the contrary, the minimum cross section lies between the disc and the ring.) The case of motion of the disc for $z_1 > C_2(\varepsilon)$ is not considered here.

Relations (1.3) define two families of meridians (1.1) of the catenoid.

In Fig. 2 for $\varepsilon = 0.1$ and $z_1^\pm = 0.01$ the dashed curve plots the meridian of the first family (1.1) (the plus sign in expressions (1.3)) with parameters $C_1^+ = 7.408 \cdot 10^{-4}$ and $C_2^+ = 5.853 \cdot 10^{-3}$, and the solid curve plots the meridian of the second family $C_1^- = 4.342 \cdot 10^{-3}$, $C_2^- = 2.663 \cdot 10^{-2}$. (The case of non-uniqueness of the extremal was noted, for example, in the problem of the brachistochrone.[2])

From a physical point of view, deformation of the membrane should depend continuously on the z coordinate of the disc centre z_1. The types of the meridians are preserved in the region of definition of z_1. Numerical examples confirm that the surface area of a truncated catenoid of the second type (a catenoid bounded by the $z = 0$ and $z = z_1$ planes)

$$S_1 = \pi C_1 z_1 + \frac{1}{2}\pi C_1^2\left[\mathrm{sh}\left(2\frac{z_1 - C_2}{C_1}\right) + \mathrm{sh}\left(2\frac{C_2}{C_1}\right)\right] \tag{1.5}$$

is less than the surface area of a truncated catenoid of the first type. It can be represented with arbitrarily assigned accuracy in the form of a polynomial in z_1 on the interval $[0, C_1^-]$. Expression (1.5) has an indeterminacy at $z_1 = 0$. Using the standard method it is possible to show that $S_1(0) = \pi(1 - \varepsilon^2)$.

Motion of the disc on the membrane is described by the energy integral

$$\dot{z}_1^2 = \dot{z}_0^2 - 2\mu\Delta S_{10}, \quad \Delta S_{10} = S_1 - S_1(0), \quad \dot{z}_0 = \dot{z}(0) \tag{1.6}$$

where ΔS_{10} is the area increment of the membrane with respect to the unperturbed state and $\dot{z}_0 = \dot{z}(0)$ is the disc velocity before the collision. In Fig. 3 the dashed curve plots the function $s_{10} = \Delta S_{10}/\pi$ for $\varepsilon = 0.1$.

Defining the smallest positive root z_{min} of the equation $\dot{z}_0^2 - 2\mu\Delta S_{10} = 0$ (for example, from the graph in Fig. 3), we find the maximum sag of the membrane.

If the right hand in Eq. (1.6) is represented in the form of a polynomial of fourth degree in z_1, the solution of this equation can be expressed in terms of Weierstrass functions.[3] Considering s_{10} as a function of the parameter C_1, we obtain the phase trajectory of the disc.