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# Waves in one-dimensional quasicrystalline structures: dynamical trace mapping, scaling and self-similarity of the spectrum

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ABSTRACT

Harmonic axial waves in quasiperiodic-generated structured rods are investigated. The focus is on infinite bars composed of repeated elementary cells designed by adopting generalised Fibonacci substitution rules, some of which represent examples of one-dimensional quasicrystals. Their dispersive features and stop/pass band spectra are computed and analysed by imposing Floquet–Bloch conditions and exploiting the invariance properties of the trace of the relevant transfer matrices. We show that for a family of generalised Fibonacci substitution rules, corresponding to the so-called *precious means*, an invariant function of the circular frequency, the Kohmoto's invariant, governs self-similarity and scaling of the stop/pass band layout within defined ranges of frequencies at increasing generation index. Other parts of the spectrum are instead occupied by almost constant *ultrawide* band gaps. The Kohmoto's invariant also explains the existence of particular frequencies, named *canonical* frequencies, associated with closed orbits on the geometrical three-dimensional representation of the invariant. The developed theory represents an important advancement towards the realisation of elastic quasicrystalline metamaterials.



## 1. Introduction

Controlling waves with mechanical metamaterials is an established research field that has reached a certain degree of maturity. Two approaches are mainly followed to achieve the goal: one is based on the investigation of dispersion properties of periodic structures composed of specifically designed unit or elementary cells (Kushwaha et al., 1993; Lin, 1962; Sigalas and Economou, 1992); the other relies on mathematical transformations that dictate the local features of the metamaterial necessary, for instance, to steer waves along predetermined paths (Brun et al., 2009; Colquitt et al., 2014; 2017; Farhat et al., 2009; Maldovan, 2013; Milton et al., 2006; Norris, 2008; Parnell et al., 2012).

With reference to the first approach, a possible way to conceive the unit cell is that based on quasiperiodic sequences. These are formed by a set of –typically two– homogeneous parts combined to create non-periodic patterns which can be generally described through deterministic rules (commonly known as generation or substitution rules). Depending on the properties of these laws, two distinct classes of quasiperiodic structured media can be identified: *quasicrystalline* structures (Levine and Steinhardt, 1984) and *non-quasicrystalline* deterministic systems (Huang et al., 1992). In the one-dimensional setting, a rigorous method of classification for the different quasiperiodic patterns was proposed by Kolar (1993). Based on

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this criterion, we define a one-dimensional quasiperiodic chain composed of two distinct elements, say $L$ and $S$, generated according to the generic substitution rule

$$L \to \varsigma(L) = \mathcal{M}_{\alpha\beta}(L, S), \quad S \to \varsigma(S) = \mathcal{N}_{\gamma\delta}(L, S), \tag{1}$$

where $\mathcal{M}_{\alpha\beta}(L, S)$ and $\mathcal{N}_{\gamma\delta}(L, S)$ are two building blocks consisting of a certain permutation of $\alpha + \beta$ and $\gamma + \delta$ elements, respectively. Parameters $\alpha$ and $\beta$ denote the number of elements $L$ and $S$ in $\varsigma(L)$, respectively, whilst $\gamma$ and $\delta$ are their counterpart in $\varsigma(S)$. Introducing the structure parameter $w = \beta\gamma - \alpha\delta$, the condition for having a quasicrystalline system is $w = \pm 1$. Quasicrystalline media possess very peculiar characteristics that make them an intermediate class of structured materials between periodic ordered crystals and random media (Steurer, 2004; Steurer and Deloudi, 2008). A typical example of one-dimensional quasicrystalline pattern is represented by the Fibonacci *golden* sequence for which $\alpha = \beta = \gamma = 1$ and $\delta = 0$, while in plane problems an example of a quasicrystalline tessellation is the Penrose tiling (Penrose, 1974). Conversely, an example of non-quasicrystalline deterministic system whose properties are more similar to those of a random media is represented by the so-called Thue–Morse chain (Tamura and Nori, 1989).

The electromagnetic behaviour of one-dimensional quasicrystalline electronic, optical and magnetic media has been extensively studied both theoretically (Kohmoto et al., 1983; 1987; Kolar and Ali, 1989b) and experimentally (Laruelle and Etienne, 1988). All these investigations have shown that although quasicrystalline systems are not periodic, their features can be described using quasiperiodic approximants. Moreover, their electronic and optical spectra possess a self-similar ordered layout characterised by scaling laws which cannot be observed in periodic or purely random media (Kohmoto and Oono, 1984).

In mechanics, despite a few attempts to study dispersion properties of elastic Fibonacci-generated waveguides (Gei, 2010; King and Cox, 2007; Zhao et al., 2013), the understanding of these scaling phenomena has not yet been satisfactorily addressed for quasicrystalline and general quasiperiodic structures. An investigation is therefore required to reveal the basic features of dynamic spectra and provide the necessary guidelines for their possible exploitation in the design of novel architectured materials whose stop and pass band topology can be easily modulated and controlled.

In this paper, waves in one-dimensional phononic quasicrystalline systems for applications in structural mechanics are thoroughly studied. In particular, our goals are:

- to provide a general framework to analyse axial harmonic wave propagation of quasicrystalline generalised Fibonacci rods;
- to highlight the role of trace mapping and that of an invariant function, the Kohmoto's invariant, in determining the properties of harmonic dynamics of such structures;
- to study the scaling properties of the dynamic spectra exploiting the features of the Kohmoto's invariant;
- to investigate the occurrence of *ultrawide* stop bands occurring in the dynamic spectra;
- to introduce a special class of quasicrystalline structures, named *canonical* structures, that display special conservation properties in the stop/pass band diagram.

The outcome of this paper sets out a methodology to be applied to the mechanics of quasicrystalline-generated beams, plates and composite materials.

## 2. One-dimensional generalised Fibonacci structures

We introduce a particular class of infinite, one-dimensional, bi-component quasiperiodic structures. Its elements are composed of a repeated elementary cell where two distinct elements, say $L$ and $S$, which can be springs, rods or supported beams, are arranged in series according to the generalised Fibonacci sequence (Poddubny and Ivchenko, 2010). The repetition of such quasiperiodic fundamental cells implies global periodicity along the axis and then the possibility of applying the Floquet–Bloch technique in order to study harmonic wave propagation in these systems. The generalised two-component Fibonacci sequence is based on the following substitution rule (Kolar and Ali, 1989b):

$$L \to \varsigma(L) = L^m S^\ell, \; S \to \varsigma(S) = L, \quad \text{with } m, \, \ell \geq 1, \tag{2}$$

where the exponent indicates the times the base is repeated, i.e. $L^m = LLL\ldots$ ($m$ times). In terms of the general definition (1), the parameters of the substitutive relation (2) are given by $\alpha = m, \; \beta = \ell, \; \gamma = 1, \; \delta = 0$ and $w = \ell$. Expression (2) implies that the finite generalised Fibonacci sequence of the $i$th order ($i = 0, 1, 2, \ldots$), here denoted by $\mathcal{F}_i$, obeys the recursive rule

$$\mathcal{F}_i = \mathcal{F}_{i-1}^m \mathcal{F}_{i-2}^\ell, \quad \text{with } m, \, \ell \geq 1, \tag{3}$$

where the initial conditions are $\mathcal{F}_0 = S$ and $\mathcal{F}_1 = L$. The total number of elements of $\mathcal{F}_i$ corresponds to the generalised Fibonacci number $\tilde{n}_i$ given by the recurrence relation

$$\tilde{n}_i = m\tilde{n}_{i-1} + \ell\tilde{n}_{i-2}, \quad \text{with } i \geq 2, \tag{4}$$

and $\tilde{n}_0 = \tilde{n}_1 = 1$. The limit $\sigma$ of the ratio $\tilde{n}_{i+1}/\tilde{n}_i$ for $i \to \infty$ is

$$\sigma = \lim_{i\to\infty} \frac{\tilde{n}_{i+1}}{\tilde{n}_i} = \frac{m + \sqrt{m^2 + 4\ell}}{2}. \tag{5}$$