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Existence for constrained dynamic Griffith fracture with a weak maximal dissipation condition

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ABSTRACT

The study of dynamic fracture is based on the dynamic energy-dissipation balance. It is easy to see that this condition is always satisfied by a stationary crack together with a displacement satisfying the system of elastodynamics. Therefore to predict crack growth a further principle is needed. In this paper we introduce a weak maximal dissipation condition that, together with elastodynamics and energy balance, provides a model for dynamic fracture, at least within a certain class of possible crack evolutions. In particular, we prove the existence of dynamic fracture evolutions satisfying this condition, subject to smoothness constraints, and exhibit an explicit example to show that maximal dissipation can indeed rule out stationary cracks.

1. Introduction

A long term goal of the mathematical analysis of dynamic fracture is to formulate and study precise models for crack growth when inertial effects are taken into account. As in the quasistatic case (see Bourdin et al., 2008 and the references therein), it would be desirable to consider cracks with arbitrary shape and topology. This general framework could be used to predict a behavior such as kinking and branching, without additional hypotheses. These models should be based on as few general principles as possible, determined by the basic physics of the problem. The precise formulation of these principles should lead to a rigorous mathematical proof of all details of the models.

Let us mention that the existence of solutions satisfying all conditions considered in a specific model is a primary issue. This is not just a mathematical luxury – a formulation that prescribes too many properties runs a strong risk of not having solutions. The proof of the existence of a solution in the framework of a model, under suitable assumptions on the data, guarantees that this model has no internal contradictions. Only in this case one can use it to compute approximate solutions and then compare the predictions of the model with the outcome of experiments.

Several results have already been obtained in fracture mechanics, starting from the seminal paper by Griffith (1920), who established an energy criterion for crack stability: a crack can grow only if the elastic energy released when it grows is larger than or equal to the energy spent to produce the new portion of the crack. The infinitesimal version of this principle leads to the notions of energy release rate and of toughness. In the two-dimensional case, the former is defined as the elastic energy released per unit length of new crack, and depends on the deformation of the body and on the geometry of the crack, while

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the latter is a material property, defined as the energy per unit length dissipated in the process of crack production. These notions, which nowadays are the basis of fracture mechanics, were introduced by Irwin (1957), who also showed the relationship between energy release rate and stress intensity factors in the case of linear elasticity.

Mott (1948) extended Griffith's criterion to the case of dynamic fracture, taking the kinetic energy into account. This leads to the dynamic energy-dissipation balance: the sum of the kinetic energy and of the elastic energy at time t, plus the energy dissipated by the crack between time zero and time t, is equal to the initial energy plus the total work done between time zero and time t.

The modern approach to dynamic fracture presented in the books (Freund, 1990; Broberg, 1999; Morozov and Petrov, 2000; Slepyan, 2002; Ravi-Chandar, 2004; Anderson, 2005; Zehnder, 2012) is based on these ideas. The fracture criteria are expressed in terms of the dynamic energy release rate, which is connected with the dynamic stress intensity factors. Besides the energy dissipated for crack production, these books take also into account further sources of dissipation due to thermal effects and to the speed of the crack process.

Note, however, that the dynamic energy-dissipation balance is not sufficient to determine the evolution of a crack, since elastodynamics with a stationary crack will always satisfy energy balance. This shows that there is a need for an additional principle for crack growth.

Recently, a phase-field approach has been developed, for which existence has been shown (see Bourdin et al., 2008, 2011; Bouchbinder et al., 2010; Larsen et al., 2010; Hofacker and Miehe, 2013). The crack is replaced by a phase-field approximation: a function v which takes the value 0 near the crack and the value 1 far from it. In these models, an energy minimization condition on v provides a principle that can require the crack to grow (so that stationary cracks are not always solutions).

Although this approach is very efficient from the numerical point of view, the convergence of the phase-field solutions to the solutions of the crack problem as the approximation parameter tends to 0 has been proved only in the quasistatic regime. Indeed, the correct form of the limiting problem, with a sharp crack instead of a regularized crack, is not yet completely clear. One possible principle to replace the minimality in the phase-field model is a maximal dissipation condition that does not require any crack regularity, proposed in Larsen (2010).

The aim of this paper is to introduce a variant of this maximal dissipation condition and to show that, under suitable assumptions on the initial and boundary conditions, we can prove the existence of a solution to the dynamic crack problem with a prescribed crack path. This is defined as a crack-displacement pair such that the displacement satisfies the system of elastodynamics out of the crack set and the pair satisfies the dynamic energy-dissipation balance and the maximal dissipation condition. We also show that this condition is strong enough to prevent static cracks from being solutions in the case of certain initial and boundary conditions.

To our knowledge, this is the first mathematical proof of an existence result for dynamic fracture with sharp cracks. Since the interaction between elastodynamics and crack growth is a source of severe mathematical difficulties, we attack this problem in a simplified model, which still exhibits some of the relevant mathematical difficulties. The long term goal is to extend these results to the fully general problem – the dream is to avoid any assumptions about the regularity or any other properties of the crack set.

The model we consider here is linearly elastic with antiplane displacement. Therefore, the reference configuration is contained in the plane, the displacement u is scalar, and the system of elastodynamics reduces to the scalar wave equation. The crack follows a sufficiently regular prescribed path. We consider only the problem of crack growth, assuming that an initial crack is already present. The issue of crack nucleation is out of the scope of this paper.

We neglect all thermal effects, as well as other sources of dissipation, except for the energy spent to produce new crack. Our point is that the main mathematical difficulties to obtain an existence result are already present in this simplified model, and that more realistic models could be studied later by adapting the ideas and techniques developed here.

As described in detail below, the crack evolutions we are considering are characterized by their length $s(t)$, considered as a function of time t. Given such a (sufficiently regular) function s, and appropriate initial and boundary data, to find the corresponding displacement u we have to solve the wave equation off the time-dependent crack set, with zero Neumann condition on the crack.

This problem already exhibits a mathematical difficulty, due to the time dependence of the domains. The classical results on the wave equation in time-dependent domains cannot be applied directly, since they require suitable regularity assumptions on the boundary, which are clearly not satisfied by cracked domains.

The existence of a solution to this problem in domains with a prescribed growing crack was proved in Dal Maso and Larsen (2011) under very general assumptions, which allow for kinking and branching. The uniqueness, however, is an open problem in this general setting. Since in our treatment of the problem the uniqueness of the solution u of the wave equation off the crack set is crucial, in our model with a prescribed crack path we assume more regularity on s in order to apply the uniqueness result proved in Dal Maso and Lucardesi (2015). In particular, we assume some uniform bounds on the first three time derivatives of s.

According to Mott's ideas, in our model we assume the dynamic energy-dissipation balance. Using the standard rule to compute the dynamic energy release rate for mode III cracks, it is possible to show that, under appropriate regularity assumptions, for every crack-displacement pair satisfying the dynamic energy-dissipation balance the dynamic stress intensity factor must be proportional to the toughness when the crack tip moves with a positive speed, with a universal proportionality constant. We will not develop this topic in this paper.

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