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On asymptotic elastodynamic homogenization approaches for periodic media

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ABSTRACT

A fairly large family of asymptotic elastodynamic homogenization methods is shown to be derivable from Willis exact elastodynamic homogenization theory for periodic media under appropriate approximation assumptions about, for example, frequencies, wavelengths and phase contrast. In light of this result, two long-wavelength and low-frequency asymptotic elastodynamic approaches are carefully analyzed and compared in connection with higher-order strain-gradient media. In particular, these approaches are proved to be unable to capture, at least in the one-dimensional setting, the optical branches of the dispersion curve. As an example, a two-phase string is thoroughly studied so as to illustrate the main results of the present work.

1. Introduction

For the elastodynamic homogenization of periodically inhomogeneous media, a large family of methods based on asymptotic analysis (see, for example, Daya et al., 2002; Auriault and Bonnet, 1985; Boutin and Auriault, 1993; Craster et al., 2010; Nolde et al., 2011; Antonakakis et al., 2014; Colquitt et al., 2014; Auriault and Boutin, 2012; Boutin et al., 2014) has been proposed and developed since the pioneer works of Bensoussan et al. (1978) and Sanchez-Palencia (1980). Making the key hypothesis of scale separation, or long wavelengths (LW), and adopting some additional assumptions such as low-frequency (LF), finite-frequency (FF), high or low phase contrast, asymptotic elastodynamic homogenization approaches consist in first expanding and computing the relevant local (or microscopic) fields term by term up to, theoretically speaking, getting an arbitrarily high accuracy, and then constructing the macroscopic fields and the effective constitutive properties by carrying out appropriate volume averages over a unit cell. Even though these approaches have been proven to be useful and efficient in many situations, two important questions remain largely open. First, apart from guiding asymptotic expansions, does the scale separation hypothesis also play a role in defining the effective fields for a periodic medium? Second, what is the effective elastodynamic behavior if the asymptotic expansions are made up to infinite order?

At the same time and independently, Willis initiated and presented an elegant elastodynamic homogenization theory for periodically and randomly inhomogeneous media (Willis, 1980a,b, 1981, 1985; Willis, 1997). Recently, owing to increasing interest in acoustic metamaterials and cloaking (see, e.g., Liu et al., 2000; Milton et al., 2006; Norris, 2008; Chen and Chan, 2010; Norris and Shuvalov, 2011; Lee et al., 2012), Willis theory has been substantially developed by him-self and other researchers (Milton and Willis, 2007, 2010; Nemat-Nasser and Srivastava, 2011, 2012, 2013; Srivastava and Nemat-Nasser, 2011; Nemat-Nasser et al., 2011; Shuvalov et al., 2011; Norris et al., 2012; Willis, 2009, 2011, 2012; Nassar et al., 2015). In

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contrast with asymptotic homogenization approaches, Willis theory is exact and does not make any scale separation hypothesis. In addition, the effective elastodynamic constitutive law obtained by Willis theory for a homogenized periodic medium produces the same dispersion relation as in the initial periodic one (see, e.g., Nemat-Nasser and Srivastava, 2011). However, Willis effective elastodynamic behavior is in general non-local both in time and space, so that its numerical determination and implementation are tough and quite expensive. Further, because of the generality of Willis theory, some supplementary conditions need being specified and imposed for its application to be physically sound. This issue has been addressed and studied in our recent work about periodic media (Nassar et al., 2015).

The principal purpose of the present work is to reveal the connections between asymptotic elastodynamic homogenization approaches and Willis relevant exact theory. By achieving this objective, we hope to gain a deep insight into them and contribute to their respective development. The main results obtained by the present work can be summarized as follows:

(i) It is demonstrated that a quite large class of asymptotic elastodynamic homogenization approaches can be derived as approximations to Willis exact theory for periodic media under appropriate assumptions on loadings, microstructure and/or phase properties. Precisely, after introducing a scaling, an asymptotic theory can be deduced by expanding Willis theory over a neighborhood defined by the scaling. The resulting asymptotic theory can in turn serve as approximating and interpreting Willis theory.
(ii) Under the LW and LF hypotheses, two asymptotic elastodynamic homogenization approaches are explicitly deduced from Willis theory so that they extend to the elastodynamic context the asymptotic elastostatic ones proposed by Boutin (1996) and Smyshlyaev and Cherednichenko (2000) to account for higher-order strain-gradient effects. In addition, for highly contrasted media or under the LW and FF assumptions, the asymptotic approaches initiated by Auriault and Bonnet (1985) and by Daya et al. (2002), respectively, are shown to be consistent with a modified Willis theory first suggested by Milton and Willis (2007).
(iii) While the effective elastodynamic constitutive behavior obtained by Willis theory generates the same dispersion relation as the initial one at the microscopic level, the LW-LF asymptotic elastodynamic homogenization methods are shown to necessarily omit, at least in the one-dimensional setting, the optical branches of the dispersion curve.
(iv) For a two-phase string, Willis theory and the LW and LF asymptotic approaches are applied, and the corresponding effective elastodynamic properties are analytically obtained and numerically discussed.

The paper is organized as follows. The second section gives a short summary for Willis theory and introduces the corresponding effective displacement and body force fields which are related by an exact effective impedance incarnating the exact effective constitutive law. The third section presents asymptotic homogenization methods in two steps. The first step consists in giving two-scale representations of fields before introducing any small parameters. This demonstrates that the resulting asymptotic theories define the same notion of effective behavior as in Willis theory. In the second step, a small parameter is introduced through what is called a scaling or an imbedding, and the corresponding asymptotic expansion is carried out. In Section 4 where the LF-LW hypothesis is adopted, Willis theory is used to derive two asymptotic homogenization approaches which extend to the elastodynamic setting the two ones proposed by Boutin (1996) and by Smyshlyaev and Cherednichenko (2000) in the elastostatic context. In Section 5, the dispersion relations in Willis theory and asymptotic theories are carefully examined and compared. In Section 6, some of the main results obtained in the preceding sections are illustrated and discussed by studying the elastodynamic behavior of a 1D two-phase string. In Section 7, a short conclusion is provided.

2. Willis' exact elastodynamic homogenization theory: a short summary

In the context of periodic media, the elastodynamic homogenization theory of Willis can be completely established through a purely spatial formulation in which the Bloch (or Floquet) wave expansions play a key role (Willis, 2011). Such a spatial formulation is adopted in what follows.

Consider a periodic medium Ω and let be given a pair of wavenumber $\boldsymbol{k}$ and frequency ω. We prescribe over Ω a harmonic plane wave body force

$$\boldsymbol{f}(\boldsymbol{x}, t) = \tilde{\boldsymbol{f}} e^{i(\boldsymbol{k}\cdot\boldsymbol{x}+\omega t)} \tag{2.1}$$

where $i = \sqrt{-1}$, $\tilde{\boldsymbol{f}}$ is a constant force vector amplitude and $(\cdot)$ stands for the usual inner product. The periodicity hypothesis implies that the resulting displacement field $\boldsymbol{u}$ in Ω is a Bloch wave of the form

$$\boldsymbol{u}(\boldsymbol{x}, t) = \tilde{\boldsymbol{u}}(\boldsymbol{x}) e^{i(\boldsymbol{k}\cdot\boldsymbol{x}+\omega t)} \tag{2.2}$$

where $\tilde{\boldsymbol{u}}(\boldsymbol{x})$ is a time-independent and spatially $\mathcal{R}$-periodic displacement amplitude with $\mathcal{R}$ representing the periodicity lattice associated to Ω.

In what follows, the time dependence will be dropped when there is no risk of confusion. In terms of $(\boldsymbol{u}, \boldsymbol{f})$, the harmonic motion equation over Ω can be written as