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Journal of Applied Research and Technology

Journal of Applied Research and Technology 14 (2016) 62–66

www.jart.ccadet.unam.mx

Original

Stability of a ring of coupled van der Pol oscillators with non-uniform distribution of the coupling parameter

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Received 7 September 2015; accepted 25 January 2016
Available online 2 March 2016

Abstract

The stability of a ring of coupled van der Pol oscillators is studied in this work considering a non-uniform distribution of the coupling parameter along the ring. The stability analysis is based on the transformation of the linearized equation of the ring into a canonical Hill equation. A stability condition is derived considering the stability of the non-periodic term of the Hill equation. Stable and unstable dynamic behavior of the ring is studied by means of numerical simulations.

Keywords: Coupled oscillators; Coupling parameter; Hill equation; Stability condition; van der Pol oscillator

1. Introduction

The van der Pol (VDP) oscillator has been studied since almost a century ago, and it is considered a classical prototype of a self-excited oscillator (Van der Pol & Van der Mark, 1928). It has been used to model oscillations in a wide variety of applications such as biological rhythms, heartbeats, chemical oscillations, electrical circuits and circadian rhythms (Barron, Medina, & Hilerio, 2014; Nana & Woafo, 2006). The study of coupled oscillators provides information on emergent properties of the coupled system, such as synchronization, clustering, oscillation death (Barron, Hilerio, & Plascencia, 2012), oscillation modes and stability (Ablowitz, 1939). A common coupling between VDP oscillators that has been examined is that between a pair of oscillators (Aggarwal & Richie, 1966; Storti & Rand, 1987; Storti & Reinhall, 2000). Studies of three coupled oscillator are not common (Bakri, Nabergoj, & Tondl, 2007), and the analysis of four coupled van der Pol oscillators has been tackled by several authors (Barron & Sen, 2009; Endo & Mori, 1978; Woafo & Enjieu, 2004). The analysis of the synchronization of a ring of four identical mutually coupled VDP oscillators by means of the Floquet theory is reported in Woafo and Enjieu (2004). The four nonlinear coupled equations of the VDP ring are linearized around the non-perturbed limit cycle and then, by introducing diagonal variables, the linearized equations are changed into a second order homogeneous linear differential equations. Thereafter, by a variable transformation, these equations are converted into a group of canonical Hill equations. Finally, the stability analysis and the synchronization of the VDP ring is carried out applying the Floquet theory. Three domains of stability are reported, and the numerical results are experimentally corroborated in a later work (Nana & Woafo, 2006).

Coupled behavior of VDP oscillators has been studied in the past considering different conditions and applying diverse analysis techniques, for example: diffusive displacement coupling (Aggarwal & Richie, 1966); weak displacement coupling using harmonic balance (Linkens, 1976); weak displacement and velocity couplings using perturbation methods (Rand & Holmes, 1980); strong coupling with detuning using perturbation methods (Storti & Rand, 1982); strong diffusive coupling using matched asymptotic expansions (Storti & Rand, 1986); weak and moderate amplitude coupling with numerical solution (Pastor, Perez, Encinas, & Guerra, 1993); weak and strong bath coupling using the Floquet theory and numerical solution

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Peer Review under the responsibility of Universidad Nacional Autónoma de México.

http://dx.doi.org/10.1016/j.jart.2016.01.002

(Camacho, Rand, & Howland, 2004; Woafo & Enjieu, 2004); Whittaker method to determine the synchronized states in a ring of four mutually coupled nonidentical van der Pol oscillators (Nana & Woafo, 2006).

The collective dynamics of large rings of coupled VDP oscillators has been previously analyzed by some of the authors of the present work (Barron et al., 2014; Barron & Sen, 2013; Barron, Sen, & Corona, 2008). In Barron et al. (2008) the stability and synchronization of a large ring of N coupled identical VDP oscillators under constant coupling parameter distribution is analyzed. In Barron and Sen (2013) the effect of the singularity of the coupling matrix on the ring dynamics is explored. When this becomes singular, an infinite number of steady states is present, and the phenomenon of oscillation death arises. In Barron et al. (2014) the collective behavior of a ring of coupled identical VDP oscillators is numerically studied under constant, Gaussian and random distributions of the coupling parameter. Single and multiple coupled frequencies are obtained using the power spectra of the long term time series. In spite of this, many aspects remain to be clarified for large rings of coupled VDP oscillators.

In this work the results reported in Woafo and Enjieu (2004) and Barron et al. (2008) are extended to the analysis of the stability and synchronization of a large ring of N coupled VDP oscillators under a sinusoidal distribution of the coupling parameter. Numerical simulations show that for a ring up to one hundred oscillators synchronization and stability are feasible whenever a derived stability condition is satisfied.

2. Mathematical model

The van der Pol oscillator is mathematically expressed as (Guckenheimer, Hoffman, & Weckesser, 2003; Van der Pol & Van der Mark, 1928)

$$\ddot{x} + a(x^2 - 1)\dot{x} + x = 0 \tag{1}$$

where x is the oscillator position and a is the oscillator constant. In this work, the case of a ring of VDP oscillators in which each oscillator is coupled to its two nearest neighbors is considered. This ring is depicted in Figure 1 (Barron & Sen, 2013). The following expression arises for a ring of N oscillators:

$$\ddot{x}_i + a(x_i^2 - 1)\dot{x}_i + x_i = b_i(x_{i-i} - 2x_i + x_{i+1}) \tag{2}$$

where $1 \le i \le N$; b_i is the coupling parameter corresponding to the ith oscillator. If θ_i is the angular position of the ith oscillator, then $\theta_i = i\Delta\theta$, where $\Delta\theta = 2\pi/(N-1)$.

A non-uniform distribution of b_i is considered here. Particularly, in the present work, a sinusoidal distribution of b_i is assumed:

$$b_i = B_b + A_b \sin\left[(i-1)\frac{2\pi}{N-1}\right] \tag{3}$$

where B_b is a reference value and A_b is the amplitude of the sinusoidal distribution of b_i.

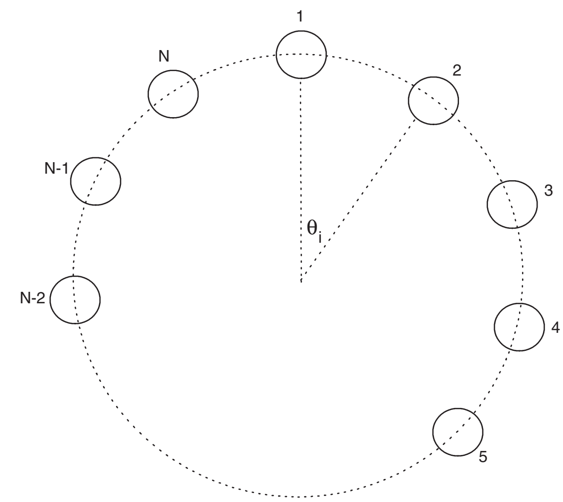

Fig. 1. Ring of coupled Van der Pol oscillators (Barron & Sen, 2013).

3. Stability analysis

To analyze the ring stability, the methodology employed in Woafo and Enjieu (2004) and Barron et al. (2008) is applied. Full description of this methodology can be found in Barron et al. (2008), then just the main results are described and extended here. To linearize Eq. (1) it can be assumed that

$$x(t) = \xi(t) + x_0(t) \tag{4}$$

where, for small values of the oscillator constant

$$x_0(t) = A\cos(\omega t - \phi) \tag{5}$$

In the above equation A is the oscillation amplitude, t is time, ω is the angular frequency and ϕ is the phase shift. Besides, the following diagonal variables can be defined:

$$z = \sum_{i=1}^{N} \delta_i \xi_i \tag{6}$$

where $\delta_i = 1$ if i is even and $\delta_i = -1$ if i is odd.

If the diagonal variables are substituted into the linearized equation of the ring, a second order homogeneous linear differential equation is obtained:

$$\ddot{z}_i + a_{1i}(\tau)\dot{z}_i + a_{0i}(\tau)z_i = 0 \tag{7}$$

where $\tau = \omega t - \phi$.

A change of variable defined by (Rand & Holmes, 1980)

$$z_i(\tau) = y_i(\tau)e^{-\frac{1}{2}\int_0^{\tau} a_{1i}(s)ds} \tag{8}$$

is applied to Eq. (7), then this equation is transformed into another one with canonical form of the Hill equation (Magnus & Winkler, 1966)

$$\ddot{y}_i + p_i(\tau)y_i = 0 \tag{9}$$