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Uniqueness of inverse problems of isotropic incompressible three-dimensional elasticity

Uri Albocher [a], Paul E. Barbone [b], Assad A. Oberai [c], Isaac Harari [a,*]

[a] Faculty of Engineering, Tel Aviv University, 69978 Ramat Aviv, Israel
[b] Department of Mechanical Engineering, Boston University, 110 Cummington St., Boston, MA 02215, USA
[c] Department of Mechanical, Aerospace and Nuclear Engineering, Rensselaer Polytechnic Institute, Troy, NY 12180, USA

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ABSTRACT

The uniqueness of an inverse problem of isotropic incompressible three dimensional elasticity aimed at reconstructing material modulus distributions is considered. We show that given a single strain field and no boundary conditions, arbitrary functions may have to be prescribed to make the solution unique. On the other hand, having two linearly independent strain fields leads to a favorable solution space where a maximum of five arbitrary constants must be prescribed to guarantee a unique solution. We solve inverse problems with two strain fields given using the adjoint weighted equation method and impose five discrete constraints. The method exhibits good numerical performance with optimal rates of convergence.

1. Introduction

In elastography (Ophir et al., 1991; Céspedes et al., 1993; Belaid et al., 1994; Ophir et al., 1996; Garra et al., 1997; Konofagou and Ophir, 2000; Konofagou et al., 2002; Hoyt et al., 2005; Righetti et al., 2007), images of tissue stiffness ("elastograms") are created for the purpose of medical diagnosis. The underlying assumption is that tissue pathology often gives rise to changes in tissue stiffness. One of the most important applications in which elastography can be used is tumor detection. In many cases, tumors are significantly stiffer than their surrounding tissues, making them stand out clearly in elastograms. In addition, differentiation between tumor types (e.g. benign and malignant) may be also possible by quantitatively monitoring changes in tissue stiffness (Samani et al., 2007). Recently, elastography has also been used to map the stiffness of cells with sub-cellular precision (Canović et al., 2013), and to determine the heterogeneous mechanical properties of hydrogels (Oberai et al., 2004; Richards et al., 2009).

Generation of elastograms involves two stages. First, tissue response to loads is measured. This is often done be applying gentle quasistatic compression directly to the tissue, and measuring the resulting displacement field in the region of interest (the area where tissue stiffness is to be evaluated). The second stage is to find the distribution of tissue stiffness. This usually involves solving an inverse problem of elasticity, where the measured displacement field is used as the input.

Solving the inverse problem of elasticity can be very challenging. One of the main challenges is related to inaccuracies in the measured displacement fields. A common method to measure displacements is using ultrasound techniques (Richards

* Corresponding author. Tel.: +972 3 6409439.
E-mail addresses: urialbocher1@gmail.com (U. Albocher), barbone@bu.edu (P.E. Barbone), oberaa@rpi.edu (A.A. Oberai), harari@eng.tau.ac.il (I. Harari).
URLS: http://www.bu.edu/me/people/faculty/ae/barbone.html (P.E. Barbone), http://www.rpi.edu/~oberaa/ (A.A. Oberai), http://www.eng.tau.ac.il/~harari (I. Harari).

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et al., 2009). While ultrasound equipment is relatively inexpensive and widely available, it can only provide an accurate measurement of the displacement field along the axis of the transducer. In the plane normal to the transducer the measurements are typically much less accurate and therefore are not very useful as data for determining tissue stiffness. Another option to measure tissue displacement is using MRI equipment (Muthupillai et al., 1995). While MRI equipment can provide accurate measurements in all directions, it is also much more expensive and therefore less commonly used.

Even when an accurate displacement field is available, the inverse problem cannot be solved without auxiliary information about the tissue stiffness (the solution). The starting point for analyzing the solvability of the inverse elasticity problem is to interpret the equations of motion as a system of partial differential equations (PDEs) where strains appear as known fields. The unknowns are the material property distributions, and for the incompressible case, the pressure distribution. In most cases these PDEs for the material property distributions are hyperbolic. However, they are not supplemented with any boundary data because the material property distribution on the boundaries is also unknown. Without boundary conditions solutions to PDE's involve arbitrary *functions*, which must be prescribed to guarantee uniqueness. This is often also the case for the inverse problem of elasticity and thus, other means to constrain the solution are necessary. One way to achieve this is by generating additional differential equations for the *same* unknown material property distribution. These equations can be obtained by taking additional measurements of displacements, that arise from different loads applied to the *same* tissue. Using these additional measurements, the necessary boundary conditions can be replaced by much more simple constraints that involve very little information about the solution. Instead of arbitrary functions, the solution would involve arbitrary *constants* only.

Recently, the uniqueness of several inverse problems of plane elasticity have been considered. The inverse problem of incompressible plane stress elasticity for example, involves a system of two equations solved for a single unknown (e.g. the shear modulus). In this case, a single displacement field is sufficient to obtain a solution involving a single arbitrary constant only (Barbone and Oberai, 2007). This is also the case for the problem of compressible plane strain elasticity where only one of the two Lamé parameters is unknown. If both Lamé parameters are unknown, a single displacement field provides a solution which is too general (involving arbitrary functions). With two displacement fields on the other hand, the most general solution would involve two arbitrary constants at most (Barbone et al., 2010). For the inverse problem of incompressible plane strain elasticity two displacement fields are also necessary, but here the solution may involve up to four arbitrary constants for the shear modulus (Barbone and Gokhale, 2004). Finally, it has been demonstrated that for inverse problems involving hyperelasticity, more data (displacement fields) are needed for the incompressible plane strain case than for incompressible plane stress (Ferreira et al., 2012). These results are consistent with those of the linear problem.

In this work we consider the inverse problem of isotropic incompressible three-dimensional elasticity. We show that with a single displacement field, the most general solution to the inverse problem involves arbitrary functions. However, with the introduction of an additional displacement field, that gives rise to a linearly independent strain field, the solution space can be reduced to that involving five arbitrary constants at most. This can be compared to the problem of incompressible plane strain elasticity, where two stain fields reduce the solution space to one involving four constants. We also consider several inclusion problems where the shear modulus is reconstructed from two linearly independent strain measurements. We solve the inverse problem directly using the adjoint weighted equation (AWE) method. The AWE method is a novel formulation for direct solutions to inverse problems, first proposed for problems of thermal conductivity (Barbone et al., 2007), and later extended to general problems of elasticity (Albocher et al., 2009; Barbone et al., 2010). We show that by imposing five discrete constraints for the five arbitrary constants associated with the most general solution, the shear modulus can be reconstructed successfully using the AWE method.

2. Formulation

Consider the equilibrium equations of elasticity in the absence of body forces:

$$\nabla \cdot \boldsymbol{\sigma} = 0 \tag{2.1}$$

Here $\boldsymbol{\sigma}$ is the Cauchy stress tensor, which, for an incompressible isotropic material satisfies a constitutive relation:

$$\sigma_{ij} = -p\delta_{ij} + 2\mu\epsilon_{ij} \tag{2.2}$$

Here p represents the hydrostatic pressure, μ is the shear modulus and $\boldsymbol{\epsilon}$ is the strain tensor. Latin indices i and j take on values 1, 2, and 3. This definition for the stress tensor arises from the mixed formulation of compressible elasticity capable of representing the incompressible limit (Hughes, 2000). With this definition, the equilibrium equation becomes

$$-\nabla p + 2\nabla \cdot (\mu\boldsymbol{\epsilon}) = \mathbf{0} \tag{2.3}$$

The standard problem of incompressible elasticity for displacements is subject to the constraint of preservation of volume, namely $\mathrm{tr}(\boldsymbol{\epsilon}) = 0$.

Consider the following *inverse* problem of isotropic incompressible elasticity: Given a strain field $\boldsymbol{\epsilon}$ in $\Omega \subset \mathbb{R}^3$, find $\mu(\boldsymbol{x})$ and $p(\boldsymbol{x})$ satisfying Eq. (2.3) in Ω. Here, in contrast to the standard "forward" elasticity problem, the strains $\boldsymbol{\epsilon}$ are *given* data arising from measurements. Note that compared to the forward problem of elasticity, which is governed by a second order elliptic partial differential equation for the displacements, the inverse problem is governed by a first-order equation for the shear modulus and pressure resembling that of advection.