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Variational principles and the related bounding theorems for bi-modulus materials

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ABSTRACT

Variational principles for materials with smooth constitutive laws have been well established in literature. For materials with non-smooth constitutive relations, however, the corresponding development has not reached a mature stage. In the present paper, a series of variational principles and the related bounding theorems for bi-modulus materials, which have different elastic properties in tension and compression, are established. Illustrative examples provided demonstrate the application of the presented theoretical results.

1. Introduction

Classical theory in solid mechanics often assumes that the considered material has the same elastic properties in tension and compression. Bi-modulus materials, which exhibit asymmetric properties under tension and compression; however, are ubiquitous in real world engineering applications. For example, important engineering materials such as concrete, ceramics, rubber and some composite materials all have different tension and compression elastic moduli (Ambartsumyan, 1986). It was also reported that the ratio between compressive modulus to tensile modulus of some fiber glass-reinforced polymide can reach up to seven even at room temperature (Ambartsumyan, 1986). Both monotonic and cyclic tension and compression tests of NiTi shape memory alloys provided evidence that it has different responses in tension and compression (Liu et al., 1998). Similar phenomena were also observed for aged single crystal and polycrystalline NiTi alloys (Gall et al., 1999). Some engineering structures such as ropes and thin membranes widely used in aerospace applications can only transfer or carry tensile loads since any small compressive loads may inevitably lead to local buckling or wrinkling (Roddeman et al., 1987a, 1987b; Mosler and Cirak, 2009. Also see Fig. 1 for reference). Deployable tensegrity structures, which have found wide applications in civil engineering, are also composed of structural components (i.e., ropes and struts) that behave totally different in tension and compression (Tibert, 2002; Sultan, 2009). Moreover, a damaged brittle material containing microcracks or microvoids, generally exhibits different stiffnesses under tension and compression (Mazars et al., 1990; Chaboche, 1992; Costa-Mattos et al., 1992. Also see Fig. 2 for reference). Some bio-materials constituted by soft organic and hard inorganic materials may also exhibit bi-modulus effective constitutive behavior at macroscopic

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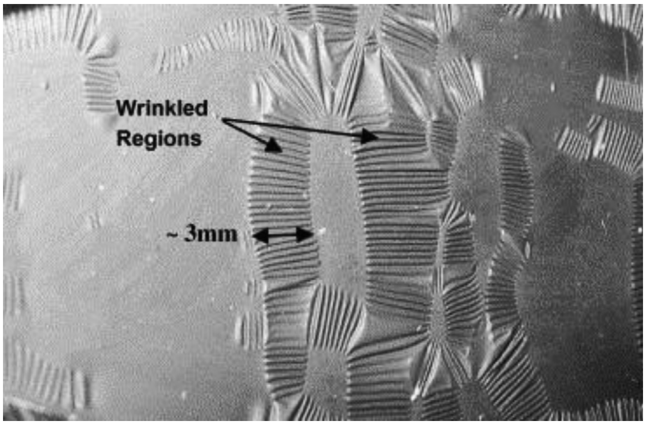

Fig. 1. Wrinkling of a dielectric elastomer film. (Plante and Dubowsky, 2006).

length scale (Bertoldi et al., 2008). Furthermore, due to the possible unilateral interlocking between asperities, the interface stiffness of rough surfaces may also exhibit sliding direction-dependent bi-modulus properties as shown in Fig. 3 schematically.

It is well-known that variational principle is a powerful tool and plays a very important role in mechanics. Actually, many governing differential equations or local physical laws used for modeling physical phenomena can all be re-casted in a variational form (Oden and Reddy, 1976; Berdichevsky, 2009) and described by the corresponding variational principles. Embedding the mechanics problems into a variational framework is also a key step to study the existence and uniqueness of solutions, establish the upper/lower bounds of overall effective properties of heterogeneous materials and develop error-controllable numerical methods. Therefore establishing variational principles for different kinds of mechanics problems is an enduring research topic in the field of theoretical and applied mechanics.

Classical variational principles for materials with smooth constitutive relations (e.g., Hooke elastic law) have been well documented in literature (Oden and Reddy, 1976; Lanczos, 1970). With the fast development of composite materials and micromechanics, variational principles had undergone a rapid expansion since the last 60's. For example, the celebrated Hashin–Shtrikman (H–S) variational principles (Hashin and Shtrikman, 1962a, 1962b, 1963) and the relevant bounding techniques have been established and well studied by many authors from different aspects (Walpole, 1966, 1969; Willis, 1977; Milton and Kohn, 1988; Milton, 1990; Allaire and Kohn, 1993a, 1993b; Murat and Tartar, 1997; Milton, 2002). Extending the Hashin-Shtrikman variational principles to allowing for materials with nonlinear properties starts from the seminal works of Talbot and Willis (1985, 1987) and Willis (1989, 1991). Later on, based on the Talbot-Willis variational framework, Ponte Castañeda (1991, 1992a) developed a series of new variational principles, which enables the description for effective energy potentials of nonlinear composites in terms of the corresponding energy potentials for linear composites. Subsequent studies (Talbot and Willis, 1992, 1994, 1997; Ponte Castaneda and Willis, 1999) showed that Talbot–Willis variational principle is also a very powerful tool to bound and estimate the effective properties of nonlinear composites.

For materials with non-smooth constitutive relations (e.g., bi-modulus material), however, the corresponding development has seldom to be found. This is partially due to the fact that the constitutive law of bi-modulus materials is dependent on the signs of principal stresses and therefore difficult to express in a unified mathematical form (Ambartsumyan, 1965; Ambartsumyan and Khachatryan, 1966; Ambartsumyan, 1986). Some other treatments of constitutive laws of bi-modulus materials could be found in (Jones, 1977; Bert, 1977; Vijayakumar and Rao, 1987; Curnier et al., 1995). The complex and non-smooth nature of the constitutive behavior of bi-modulus materials propose great challenge to construct the corresponding variational principles. Recently, Zhang et al. (2011, 2013) proposed a so-called parametric variational principle for finite element analysis of bi-modulus truss and tensegrity structures. Although rendering the construction of effective numerical solution algorithms, their variational principle has the problem that the proposed energy functional does not represent the true total potential energy of the structure. This makes it impossible to estimate the finite element solution errors and develop the corresponding adaptive mesh refinement strategies based on this energy functional. A remarkable work is in Kanno (2011), where variational principles for masonry (with zero tensile stiffness) and membrane (with zero compressive stiffness) are established with use of the tools from convex analysis.

To the best of authors' knowledge, mathematically consistent variational principles for boundary value problems involving bi-modulus materials have not been developed in literature. Furthermore, since the explicit form of the energy potential for bi-modulus materials remains elusive, it is also very difficult to construct the corresponding bounding theorems for effective overall properties of heterogeneous materials with bi-modulus components. Inspired by the work in Kanno (2011), the main purpose of the present work is to establish a series of variational principles and the related bounding theorems for bi-modulus materials.