Journal of the Mechanics and Physics of Solids 71 (2014) 46–63

Contents lists available at ScienceDirect

Journal of the Mechanics and Physics of Solids

journal homepage: www.elsevier.com/locate/jmps

Rigorous bounds on the effective moduli of composites and inhomogeneous bodies with negative-stiffness phases

Dennis M. Kochmann [a,*], Graeme W. Milton [b]

[a] Graduate Aerospace Laboratories, California Institute of Technology, Pasadena, CA 91125, USA
[b] Department of Mathematics, The University of Utah, Salt Lake City, UT 84112, USA

ARTICLE INFO

Article history:
Received 16 January 2014
Received in revised form
15 June 2014
Accepted 21 June 2014
Available online 5 July 2014

Keywords:
Stability
Elasticity
Composite
Negative stiffness
Effective properties

ABSTRACT

We review the theoretical bounds on the effective properties of linear elastic inhomogeneous solids (including composite materials) in the presence of constituents having non-positive-definite elastic moduli (so-called negative-stiffness phases). Using arguments of Hill and Koiter, we show that for statically stable bodies the classical displacement-based variational principles for Dirichlet and Neumann boundary problems hold but that the dual variational principle for traction boundary problems does not apply. We illustrate our findings by the example of a coated spherical inclusion whose stability conditions are obtained from the variational principles. We further show that the classical Voigt upper bound on the linear elastic moduli in multi-phase inhomogeneous bodies and composites applies and that it imposes a stability condition: overall stability requires that the effective moduli do not surpass the Voigt upper bound. This particularly implies that, while the geometric constraints among constituents in a composite can stabilize negative-stiffness phases, the stabilization is insufficient to allow for extreme overall static elastic moduli (exceeding those of the constituents). Stronger bounds on the effective elastic moduli of isotropic composites can be obtained from the Hashin–Shtrikman variational inequalities, which are also shown to hold in the presence of negative stiffness.

1. Introduction

The overall or effective properties of heterogeneous solids are uniquely linked to the properties of each composite constituent, their geometric arrangement and bonding. Owing to microstructural randomness in the arrangement of composite phases, effective physical properties in most cases cannot be determined exactly. One approach is to estimate them by the aid of rigorous upper and lower bounds. The simplest such bounds were introduced by Hill (1952) and Paul (1960): the Reuss and Voigt bounds are solely based on phase volume fractions and present upper and lower bounds on the effective linear elastic moduli of multi-phase composites. For inhomogeneous bodies the analogous bounds were obtained by Nemat-Nasser and Hori (1993) and by Willis in a 1989 private communication to Nemat-Nasser and Hori. Based on variational principles and the introduction of a polarization field, Hashin and Shtrikman (1963) derived new tighter bounds for isotropic well-ordered two-phase composites with bulk moduli $\kappa_2 > \kappa_1$ and shear moduli $\mu_2 > \mu_1$. Their bounds on the effective bulk modulus can be attained e.g. by assemblages of coated spheres (interchanging the materials in spherical inclusions and coatings yields upper and lower bounds on the effective bulk modulus). Similarly, hierarchical laminate

* Corresponding author. Tel.: +1 626 395 8113; fax: +1 626 395 2900.
E-mail addresses: kochmann@caltech.edu (D.M. Kochmann), milton@math.utah.edu (G.W. Milton).

http://dx.doi.org/10.1016/j.jmps.2014.06.010
0022-5096/

constructions have been shown to attain the bounds on the effective shear modulus (Norris, 1985; Milton, 1986; Francfort and Murat, 1986). Therefore, the bounds of Hashin and Shtrikman (1963) are optimal and present the strongest possible restrictions on the elastic moduli of well-ordered multi-phase solids based only on volume fractions. For non-well-ordered isotropic two-phase composites (with bulk moduli $\kappa_2 > \kappa_1$ and shear moduli $\mu_2 < \mu_1$) the tightest known bounds on the effective bulk modulus are those of Hill (1963a), and are attained by coated-sphere assemblages, and on the effective shear modulus are those of Milton and Phan-Thien (1982), which improve upon those of Walpole (1966) and are attained in certain parameter regimes where they coincide with the Hashin–Shtrikman formulae. By including statistical microstructural information of random composites, three-point bounds were derived e.g. by Beran and Molyneux (1966) and McCoy (1970), who used classical variational principles. For two-phase composites these bounds were simplified by Milton (1981). By improving McCoy's bounds, Milton and Phan-Thien (1982) found stronger restrictions for the effective shear modulus. We refer to Cherkaev (2000), Torquato (2002), Allaire (2002), Milton (2002) and Tartar (2010) for comprehensive reviews of composite bounds. Alternatively, estimates of effective composite properties have been established by an effective medium strategy, which has resulted in, among others, the self-consistent method (Hill, 1965; Budiansky, 1965; Berryman, 1980) and its generalized form (Christensen and Lo, 1979), the differential (Roscoe, 1952, 1973; Norris, 1985) and Mori–Tanaka schemes (Mori and Tanaka, 1973; Benveniste, 1987). Although beyond the scope of the present investigation, we note that nonlinear variational bounds on composite properties are available as well, see e.g. Talbot and Willis (1985), Ponte Castaneda (1991) and Castaneda and Willis (1999).

All of the aforementioned bounds imply that the effective linear elastic moduli of composites (in particular the Young, bulk, and shear moduli of isotropic composites) are bounded from above by the individual moduli of the constituent materials; i.e. no composite can be stiffer than its stiffest constituent. This prohibits the creation of new composites with extreme properties (where by '*extreme*' we refer to properties which exceed those of the constituents). However, the derivation of those bounds assume that all constituent materials possess positive-definite elastic moduli (for the specific case of isotropic solids, this is equivalent to requiring Young, bulk and shear moduli to be positive). Lakes and Drugan (2002) showed that relaxing this assumption by allowing for non-positive-definite elastic moduli (so-called negative stiffness) in one of the phases in an inhomogeneous body may lead to extreme effective stiffness. Based on exact solutions for a coated-sphere two-phase solid, they showed that a two-phase inhomogeneous body can, in principle (within the validity of the elasticity model), attain unbounded effective bulk stiffness if the constituent moduli and volume fractions are appropriately tuned. Lakes and coworkers demonstrated that various other effective physical composite properties promise to reach extreme values when including a negative-stiffness phase (Lakes, 2001a, 2001b; Wang and Lakes, 2001, 2004, 2005). Experimentally, negative stiffness has been realized by constituents undergoing microscale instabilities such as phase transitions, see e.g. Lakes et al. (2001), Jaglinski et al. (2006), Jaglinski and Lakes (2007), and Jaglinski et al. (2007). Similarly, on a structural level the negative-stiffness effect has been realized by buckling instabilities, see e.g. Moore et al. (2006), Lee et al. (2007), Lee and Goverdovskiy (2012), and Kashdan et al. (2012).

While negative stiffness is generally unstable in homogeneous solids with mixed or pure-traction boundary conditions (Kirchhoff, 1859), it was shown that negative-stiffness phases can be stabilized when geometrically constrained e.g. by a sufficiently stiff and thick coating or as inclusions in a stiff matrix (Drugan, 2007; Kochmann and Drugan, 2009, 2012; Kochmann, 2012). Unfortunately, the thus expanded stability regime is insufficient to stabilize extreme effective static stiffness in simple two-phase solids and in isotropic two-phase composites with equal shear moduli (Wojnar and Kochmann, 2014b, 2014a), while allowing for interesting dynamic phenomena.

Instead of investigating particular composite geometries, here we show that arbitrary linear elastic inhomogeneous bodies and multi-phase composites cannot reach extreme stiffness by the inclusion of negative-stiffness phases if they are to be statically stable. To this end, we first review the classical variational principles in Section 2 and determine their validity in the presence of negative-stiffness phases. We illustrate the applicability or non-applicability of the various variational principles in Appendix B by the example of a coated spherical inclusion. Next, in Section 3 we apply the variational principles to show that the classical Voigt upper bound applies and, most importantly, implies a stability condition: overall stability requires that the effective moduli must not surpass the Voigt upper bound. We further show that the Hashin–Shtrikman variational inequalities apply and that they yield additional upper and lower bounds on the effective elastic moduli of isotropic composites. Our results particularly demonstrate that extreme effective (static) elastic moduli exceeding those of any of the constituents are prohibited if the solid is to be statically stable. We further review three-point bounds on the effective moduli and conclude upper and lower bounds in the presence of negative-stiffness phases. Finally, Section 4 concludes our analysis.

2. Variational principles for heterogeneous solids

We consider an inhomogeneous body Ω containing a composite material made of elastic constituents and experiencing a displacement field $\boldsymbol{u}(\boldsymbol{x},t)$ with $\boldsymbol{x}\in\mathbb{R}^d$ denoting position in d-dimensional space and t being time. The body is subject to essential boundary conditions $\boldsymbol{u}(\boldsymbol{x})=\boldsymbol{v}(\boldsymbol{x})$ on the subset $\partial\Omega_u$ of the body's boundary $\partial\Omega$ and natural boundary conditions $\boldsymbol{t}(\boldsymbol{x})=\boldsymbol{t}_0(\boldsymbol{x})$ on $\partial\Omega_t$. Here, $\boldsymbol{t}(\boldsymbol{x})=\boldsymbol{\sigma}(\boldsymbol{x})\boldsymbol{n}(\boldsymbol{x})$ denotes the traction vector on a surface with unit outward normal $\boldsymbol{n}(\boldsymbol{x})$, and $\boldsymbol{\sigma}(\boldsymbol{x})$ is the infinitesimal symmetric stress tensor. Furthermore, we have $\partial\Omega_u\cup\partial\Omega_t=\partial\Omega$ and $\partial\Omega_u\cap\partial\Omega_t=\emptyset$. Assume that $\tilde{\boldsymbol{u}}(\boldsymbol{x})$ is the displacement field corresponding to a solution of the elasticity equation (linear momentum balance) which satisfies the above boundary conditions, i.e. $\tilde{\boldsymbol{u}}(\boldsymbol{x})$ characterizes an equilibrium configuration of the body. The arguments of Hill (1957)