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Homogenisation for elastic photonic crystals and dynamic anisotropy

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ABSTRACT

We develop a continuum model, valid at high frequencies, for wave propagation through elastic media that contain periodic, or nearly periodic, arrangements of traction free, or clamped, inclusions. The homogenisation methodology we create allows for wavelengths and periodic spacing to potentially be of similar scale and therefore is not limited to purely long-waves and low frequency.

We treat in-plane elasticity, with coupled shear and compressional waves and therefore a full vector problem, demonstrating that a two-scale asymptotic approach using a macroscale and microscale results in effective scalar continuum equations posed entirely upon the macroscale; the vector nature of the problem being incorporated on the microscale. This rather surprising result is comprehensively verified by comparing the resultant asymptotics to full numerical simulations for the Bloch problem of perfectly periodic media. The dispersion diagrams for this Bloch problem are found both numerically and asymptotically. Periodic media exhibit dynamic anisotropy, e.g. strongly directional fields at specific frequencies, and both finite element computations and the asymptotic theory predict this. Periodic media in elasticity can be related to the emergent fields of metamaterials and photonic crystals in electromagnetics and relevant analogies are drawn. As an illustration we consider the highly anisotropic cases and show how their existence can be predicted naturally from the homogenisation theory.

1. Introduction

Wave propagation through elastic periodic, or nearly periodic, media, and their simpler acoustic counterparts (Craster and Guenneau, 2012), are ripe for exploitation using recently developed ideas from photonics and metamaterials in optics. Naturally there have been studies of waves in structured elastic media, with composite fibre media providing motivation. For perfectly periodic media these can be treated using multipole techniques, plane wave expansions or using brute force numerics; the resultant dispersion diagrams (Zalipaev et al., 2002; Guenneau et al., 2003) then show stop-bands and other features highly reminiscent of those from optics. Metamaterials, and structured media such as photonic crystal fibres (Russell, 2003; Zolla et al., 2005), are topical and important subjects in optics and electromagnetic (EM) waves. The metamaterial field has expanded rapidly since Pendry showed that Veselago's convergent flat lens (Veselago, 1968) overcomes Rayleigh's diffraction limit (Pendry, 2000). This requires simultaneously negative permittivity and permeability,

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which is an essential condition for a negative refractive index, as demonstrated experimentally by Smith et al. (2000). Photonic crystals have a longer history in optics and EM (John, 1987; Yablonovitch, 1987) and also exhibit useful technological phenomena with their use now widespread (Joannopoulos et al., 2008). The area is almost exclusively preoccupied with electromagnetic (EM) waves, governed by the Maxwell system of equations: it is also clear that similar effects can occur for elasticity (Milton and Nicorovici, 2006; Bigoni et al., 2013), but the elastic case is far less developed than for optics, and EM, particularly for the vector case of coupled longitudinal-shear in-plane elastic waves. A key ingredient toward developing fundamental understanding of how a microstructured medium behaves is to upscale to an effective medium description.

The classical route to replace a microstructured medium with an effective continuum representation is homogenisation theory, and is detailed in many monographs, Sanchez-Palencia (1980), Bakhvalov and Panasenko (1989), Bensoussan et al. (1978), and Panasenko (2005), and essentially relies upon the wavelength being much larger than the microstructure which is usually assumed to be perfectly periodic: this limits the procedure to low frequencies/long waves and a quasi-static situation, cf. Parnell and Abrahams (2008) and Walpole (1992) for a general setting and Zalipaev et al. (2002) for the specific application to circular holes. Unfortunately for many real applications, particularly in photonics, the limitation to low frequencies is a critical deficiency, nonetheless the attraction of having an effective equation for a microstructured medium where one needs no longer model the detail of each individual scatterer and attention can then be given to the overall physics of the structure is highly attractive (Guenneau et al., 2012).

To overcome this limitation a high frequency homogenisation (HFH) technique has been developed, Craster et al. (2010a), to address the scenario when the wavelength and characteristic microstructural lengthscale are of similar order and multiple scattering can be important: a multiple scales' approach is utilised, a short microscale and a long macroscale, as in quasi-static homogenisation. This is augmented by utilising some knowledge from physics where it is well known that a perfectly periodic infinite medium has an exact formulation in terms of Bloch waves; moreover there are standing waves within the periodic setting that occur at discrete standing wave frequencies. The standing wave frequency and eigensolution precisely encode the multiple scattering that occurs and this additional knowledge is then built into the asymptotic procedure. The result is an effective medium theory for periodic, or nearly periodic, media that have no upper frequency limit and thus far it has successfully modelled a variety of multi-scale problems in optics (Craster et al., 2011), discrete lattice media (Craster et al., 2010b), structural mechanics (Nolde et al., 2011), elastic plates (Antonakakis and Craster, 2012) and surface Rayleigh–Bloch (acoustic or EM) waves along multi-scale surfaces (Antonakakis et al., 2013a). The theory emerged from earlier work on high frequency long-wave asymptotics for distorted waveguides (Gridin et al., 2005; Kaplunov et al., 2005), as summarised in Craster et al. (2014), and produces a continuum description posed entirely upon the long-scale, the solution of which modulates a short-scale, possibly highly oscillatory field; the long-scale equation captures the dynamic anisotropy and features associated with metamaterials and photonic crystal fibres. Thus far the HFH theory is limited to scalar wave systems such as those modelled by the Helmholtz equation, i.e. acoustics, water waves, TE or TM polarised electromagnetic waves and shear horizontal polarised elastic waves, or the extension to a fourth-order elastic plate system. Moving the entire theory to a full vector wave system has several technical and conceptual challenges and the aim of this article is to overcome them.

It is notable that the HFH theory of Craster et al. (2010a) is not alone: there is considerable motivation to create effective continuum models of microstructured media, in various related fields, that break free from the conventional low frequency homogenisation limitations. This desire has created a suite of extended homogenisation theories originating in applied analysis, for periodic media, called Bloch homogenisation (Conca et al., 1995; Allaire and Piatnitski, 2005; Birman and Suslina, 2006; Hoefer and Weinstein, 2011). There is also a flourishing literature on developing homogenised elastic media, with frequency-dependent effective parameters, also based on periodic media (Willis, 2009; Nemat-Nasser et al., 2011; Norris et al., 2012). Those approaches notwithstanding, our aim here is to extend the HFH theory previously only available for scalar Helmholtz or elastic plate cases to the full vector elastic system.

A typical situation might be a finite slab of composite medium, created from equidistant holes or fibres, through which elastic waves then propagate: a schematic is shown in Fig. 1(a) for circular holes, although the methodology is in no way limited to cylindrical defects. If the medium is of large extent, with some lengthscale L, and the holes and their spacing are of lengthscale l, with $l \ll L$, the scale mismatch suggests an asymptotic approach. On the small-scale, attention can be focused on a single elementary cell, shown dashed in Fig. 1(a), and its dynamics are then coupled into the global picture. If a medium is actually infinite and perfectly periodic then the elementary cell truly captures all the multiple scattering within a framework known as Bloch theory. Here one can assume that there is a phase shift, characterised by a wavenumber vector $\boldsymbol{\kappa}$, and from solid-state physics it is known (Brillouin, 1953; Kittel, 1996) that one needs to only focus attention on an irreducible Brillouin zone in a wavenumber space (shown in Fig. 1(b)) and this representation is useful both within the asymptotic scheme and as a verification of it. In particular, standing waves occur at the edges of the irreducible Brillouin zone (the points Γ, X, M in Fig. 1(b)) and the asymptotic technique uses the behaviour in the neighbourhood of these points to create a long-scale effective medium valid close to the corresponding standing wave frequencies.

In this paper we begin, Section 2, by formulating the two-scale approach and deriving the effective long-scale partial differential equation (PDE) for a function $f_0(\mathbf{X})$: the PDE contains integrated quantities from the short-scale, but is only posed on the long-scale. For low-frequencies we can generate a long-wave version of the theory and this is briefly described in Section 2.4. It is naturally important to verify the asymptotic technique versus full numerical simulations and we proceed to do so in Section 3. Initially, asymptotic dispersion curves for the Bloch theory are compared to full numerical simulations,