Journal of the Mechanics and Physics of Solids 71 (2014) 112–131

Contents lists available at ScienceDirect

Journal of the Mechanics and Physics of Solids

journal homepage: www.elsevier.com/locate/jmps

Shakedown of elastic-perfectly plastic materials with temperature-dependent elastic moduli

Michaël Peigney

Université Paris-Est, Laboratoire Navier (Ecole des Ponts ParisTech, IFSTTAR, CNRS), F-77455 Marne-la-Vallée Cedex 2, France

ARTICLE INFO

Article history:
Received 23 December 2013
Received in revised form
12 June 2014
Accepted 22 June 2014
Available online 28 June 2014

Keywords:
Elastic–plastic materials
Temperature-dependent material properties
Melan's theorem
Shakedown

ABSTRACT

For elastic-perfectly plastic materials with constant properties, the well-known Melan's theorem gives a sufficient condition for shakedown to occur, independently on the initial state. It has been conjectured that Melan's theorem could be extended to temperature-dependent (or time-dependent) elastic moduli, but no theoretical result is available. This paper aims at providing results in that direction, with a special emphasis on time-periodic variations. If Melan's condition is satisfied, we show that shakedown indeed occurs provided the time fluctuations of the elastic moduli satisfy a certain condition (which in particular is fulfilled if the time fluctuations are not too large). We provide a counter-example which shows that setting such a constraint on the elastic moduli is necessary to reach path-independent theorems as proposed. A simple mechanical system is studied as an illustrative example.

1. Introduction

For elastic-perfectly plastic structures under prescribed loading histories, the celebrated Melan's theorem (Melan, 1936; Symonds, 1951; Koiter, 1960) gives a sufficient condition for the evolution to become elastic in the large-time limit. That situation, classically referred to as *shakedown*, is associated with the intuitive idea that the plastic strain tends to a limit as time tends to infinity. The Melan's theorem has the distinctive property of being path-independent, i.e. independent on the initial state of the structure. For a parametrized loading history, Melan's theorem gives bounds on the domain of load parameters for which shakedown occurs. Regarding fatigue design, shakedown corresponds to the most beneficial regime of high-cycle fatigue, as opposed to the regime of low-cycle fatigue which typically occurs if the plastic strain does not converge towards a stabilized value (DangVan and Papadopoulos, 1999).

The contribution of Koiter (1960) gave rise to a lot of subsequent research, mainly along two different lines. A first line of research is concerned with extensions of the original theorem to various nonlinear behaviors, such as hardening plasticity (Pham, 2008; Nguyen, 2003), nonstandard plasticity (Corigliano et al., 1995; Bodovillé and De Saxcé, 2001), contact with friction (Ahn et al., 2008), phase-transformation (Peigney, 2010). A second (and complementing) line of research is concerned with the development of numerical methods for efficiently determining the shakedown domain in the space of load parameters (Zarka et al., 1988; Maitournam et al., 2002; Carvelli et al., 1999; Peigney and Stolz, 2001, 2003; Simon and Weichert, 2012; Spiliopoulos and Panagiotou, 2012). We refer to Weichert and Ponter (2014) for more historical details on the development of shakedown theory.

E-mail address: michael.peigney@polytechnique.org

http://dx.doi.org/10.1016/j.jmps.2014.06.008

In this paper, we aim at extending Melan's theorem to situations in which the elastic moduli are fluctuating in time. Such time fluctuations may result from significant variations of the temperature. In a lot of practical situations, structural elements are indeed submitted to thermomechanical loading histories in which variations of the temperature are large enough for the temperature dependence of the material not to be negligible. The case of the temperature-dependent yield limits has been considered by Borino (2000). In contrast, the case of temperature-dependent elastic moduli remains a long standing issue, even though the practical importance of that problem has been recognized early (König, 1969).

In order to underline the fundamental difficulties raised by temperature-dependent moduli, we observe that, for elastic–plastic materials with a constant elastic tensor, the distance between two solutions is monotonically decreasing with time (more details are given in Section 3). That property plays a crucial role in proving shakedown theorems, both in the original framework considered by Koiter as well as in most of the extensions mentioned earlier. That property does not to hold anymore for temperature-dependent elastic moduli, which indicates that temperature-dependent moduli fundamentally change the mathematical nature of the evolution problem.

Using incremental analysis, the asymptotic behavior of some specific mechanical systems has been studied by Halphen and di Domizio (2005), Hasbroucq et al. (2010), and Hasbroucq et al. (2012). The obtained results led to some conjectures about shakedown theorems for time-dependent elastic moduli, but not theoretical result is available. This paper aims at providing some results in that direction, and is organized as follows. In Section 2 we establish the differential inclusion that governs the evolution of the stress field, Eq. (15), and discuss the relation between the equation obtained and the sweeping process introduced by Moreau (1977). The main goal of the paper is to study the asymptotic behavior of the stress field, defined as a solution of Eq. (15) for some given initial condition. The well-known case of time-independent elastic moduli is briefly discussed in Section 3. We present the shakedown condition originally introduced by Koiter (1960) as well as two variations of that condition. Although those three conditions are interrelated, they are not strictly equivalent, as we shall explain. The most conclusive results are obtained when the space of residual stress fields is of finite dimension. In that case, under any of the three conditions introduced in Section 3, the stress field can be proved to converge towards an elastic response, independently on the initial state. The question is whether the same conclusions can be extended to time-dependent elastic moduli. We show in Section 4 that the general answer is no: considering time-periodic variations of the elastic moduli, we provide an explicit counterexample for which there exists an inelastic limit cycle, even though the conditions discussed in Section 3 are satisfied. For that example, the solutions of (15) are either attracted to an inelastic limit cycle or to the set of elastic solutions, depending on the initial state. This ruins any hope that the conditions of Section 3 are sufficient for shakedown to occur in a path-independent fashion. Positive results are given in Sections 5 and 6: we show that shakedown theorems can still be obtained for time-dependent elastic moduli, but their formulation differs significantly from the classical Melan's theorem. A key point in the analysis is a general bound on the variations of the elastic energy, Eq. (50), that we establish in Section 5. That result is used in Section 6 to prove some shakedown theorems when the elastic moduli are time-periodic. Those theorems are applied in Section 7 to a simple mechanical system. Numerical results of incremental analysis are also provided to illustrate the ideas discussed throughout the paper.

2. Evolution equation for the stress field

Consider an elastic-perfectly plastic material occupying a domain Ω in the reference configuration. Under the assumption of infinitesimal strains, the strain tensor $\boldsymbol{\epsilon}$ is derived from the displacement $\boldsymbol{u}$ by

$$\boldsymbol{\epsilon}=\frac{1}{2}(\nabla \boldsymbol{u}+\nabla^T \boldsymbol{u}).$$

The strain $\boldsymbol{\epsilon}$, stress $\boldsymbol{\sigma}$ and plastic strain $\boldsymbol{\epsilon}^p$ at position $\boldsymbol{x}$ and time t satisfy the constitutive equations:

$$\boldsymbol{\epsilon}(\boldsymbol{x},t)=\boldsymbol{L}(\boldsymbol{x},t):\boldsymbol{\sigma}+\boldsymbol{\epsilon}^\theta(\boldsymbol{x},t)+\boldsymbol{\epsilon}^p(\boldsymbol{x},t), \tag{1}$$

$$\dot{\boldsymbol{\epsilon}}^p(\boldsymbol{x},t)\in \partial I_{\mathcal{C}(\boldsymbol{x},t)}(\boldsymbol{\sigma}(\boldsymbol{x},t)). \tag{2}$$

In (1), $\boldsymbol{L}$ is the (symmetric positive definite) elastic moduli tensor and $\boldsymbol{\epsilon}^\theta$ is the thermal strain tensor. The double product: in (1) denotes contraction with respect to the last two indexes, i.e. $(\boldsymbol{L}:\boldsymbol{\sigma})_{ij}=\sum_{k,l}\boldsymbol{L}_{ijkl}\boldsymbol{\sigma}_{lk}$. In (2), $\mathcal{C}$ is the elasticity domain of the material, assumed to be closed and convex. The space and time dependence of $\boldsymbol{L}$, $\boldsymbol{\epsilon}^\theta$ and $\mathcal{C}$ may result from imposed variations of the temperature. For instance, the elastic properties of most materials are known to depend on the temperature θ. Such a relation can be written in the form $\boldsymbol{L}=\boldsymbol{L}(\theta)$. For imposed variations $\theta(\boldsymbol{x},t)$ of the temperature, the elastic moduli tensor $\boldsymbol{L}(\theta(\boldsymbol{x},t))$ can be regarded as a function of space and time.

The function $I_{\mathcal{C}(\boldsymbol{x},t)}$ in (2) is the indicator function of $\mathcal{C}(\boldsymbol{x},t)$ (equal to 0 in $\mathcal{C}(\boldsymbol{x},t)$, infinite otherwise) and the operator ∂ denotes the subgradient (Brézis, 1972; Rockafellar, 1997). The general definition of the subgradient is recalled for latter reference: for a function ϕ defined on a Hilbert space H, $\partial\phi$ is the multi-valued mapping defined by

$$\partial\phi(\boldsymbol{\tau})=\{\boldsymbol{a}:(\boldsymbol{\tau}'-\boldsymbol{\tau},\boldsymbol{a})_H\le\phi(\boldsymbol{\tau}')-\phi(\boldsymbol{\tau})\ \forall\boldsymbol{\tau}'\in H\} \tag{3}$$