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Initial post-buckling of variable-stiffness curved panels

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ABSTRACT

Variable-stiffness shells are curved composite structures in which the fibre-reinforcement follow curvilinear paths in space. Having a wider design space than traditional composite shells, they have the potential to improve a wide variety of weight-critical structures.

In this paper, a new method for computing the initial post-buckling response of variable-stiffness cylindrical panels is presented, based on the differential quadrature method. Integro-differential governing and boundary equations governing the problem, derived with Koiter's theory (Koiter, 1945), are solved using a mixed generalised differential quadrature (GDQ) and integral quadrature (GIQ) approach. The post-buckling behaviour is determined on the basis of a quadratic expansion of the displacement fields. Orthogonality of the mode-shapes in the expansion series is ensured by a novel use of the Moore–Penrose generalised matrix inverse for solving the GDQ–GIQ equations. The new formulation is validated against benchmark analytical post-buckling results for constant stiffness plates and shells, and compared with non-linear finite-element (FE) analysis for variable-stiffness shells. Stability estimates are found to be in good agreement with incremental FE results in the vicinity of the buckling load, requiring only a fraction of the number of variables used by the current method.

1. Introduction

Curved panels are used extensively in modern aerospace, civil, mechanical and naval engineering structures. Their excellent structural efficiency, based on high buckling loads and low mass, make them attractive solutions for lightweight structures which develop large compressive stresses. In order to take their full advantage, non-linear aspects of their behaviour are an important consideration in the design phase. In compression they often have unstable buckling modes, causing unwanted jumps in load. Furthermore, one can expect to see buckling occurring well below the calculated critical load under axial compression (Cox and Clenshaw, 1941; Jackson and Hall, 1947; Peterson, 1969; Thornburgh and Hilburger, 2005), causing linear design calculations to be non-conservative. There have been many theoretical investigations into these effects. Cox and Pribram (1948) characterised the non-linear load path of isotropic panels qualitatively and found that, in the same manner as complete cylinders (von Kármán and Tsien, 1941), a degree of unloading and shortening occurs immediately after buckling (sometimes termed “double-backing”). This behaviour is in contrast to flat plates, whose load paths are monotonic and roughly piecewise-linear. Koiter (1956) quantified the effect of curvature on a panel's stability by computing the tangent stiffness of the load path in the near vicinity of the buckling point. It was found that a small

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transition region exists in the design space, where stable post-buckling stops and unstable post-buckling begins. The numerical results of this study were confirmed by Pope (1965, 1968) who used the Principle of Virtual Work to determine approximate equilibrium paths for panels with finite-displacements and fixed mode-shapes. Fig. 1 shows Pope's results for an isotropic panel. Based on these curves it may be concluded that the effect of the curvature is to (a) significantly increase the buckling load and (b) reduce the tangent stiffness at the buckling load and hence, the stability. In practice, a panel under displacement controlled edge loading, exhibits dynamic jumping from the unstable bifurcation to the nearest stable state. This effect results in a significant drop in load.

Studies into the post-buckling behaviour of composite curved panels are much less numerous. Bauld and Khot (1982) used a Rayliegh–Ritz and finite-difference approach to compute equilibrium paths for $[0,90]_{2s}$ laminates. This was done in an attempt to replicate experimental results, but with relatively little success. Zhang and Matthews (1983) investigated the effect of a small family of stacking sequences using a non-linear Rayleigh–Ritz Fourier series formulation. Within the group of laminates analysed, bend-twist coupling was found to have a significant effect on the drop in load following bifurcation. Madenci and Barut (1994) investigated composite panels under compression using non-linear finite-element analysis, but were primarily interested in the effect of cut-outs rather than stacking sequences. They did, however, find extremely good agreement with previous experiments. More recent publications, such as Hilburger et al. (1999, 2001, 2004), have also focused on the effect of boundary conditions and cut-outs. There have, therefore, been no firm conclusions made with regard to the general effects of the stiffness constants on the post-buckling behaviour of curved panels (presumably due to the large size of the design space). If these effects are to be explored more fully, the development of efficient design tools is essential.

The design space of these structures may be increased further by assuming that the shell is manufactured with variable-angle tows (VAT) and has variable stiffness-constants. One of the first papers to introduce this concept was concerned with the optimisation of perforated composite plates for increased buckling capacity (Hyer and Lee, 1991). It was demonstrated that by forcing well supported regions of the structure to be under the highest stresses, the maximum buckling load could be increased considerably over an optimal constant-stiffness structure. It was later shown by Gürdal et al. (2008) that the use of VAT may not only increase the buckling load, but the conflicting requirements for both high in-plane stiffness and high buckling loads may be resolved. In addition to buckling optimisation, it has also been shown that VAT laminates can be used to delay first ply failure of plates under compression (Lopes et al., 2007); tune the fundamental frequency of shells (Blom et al., 2008) and increase global structural stiffness of cylindrical shells (Wu, 2008). Furthermore, advances in manufacturing techniques, such as the development of the continuous tow shearing technique by Kim et al. (2012), are making VAT designs a practical possibility. Current investigations into the post-buckling behavior of VAT structures have focused on flat plates (Rahman et al., 2011; Raju et al., 2013; Wu et al., 2013a, 2013b). The development of closed form solutions for these structures is hindered by the generality of the variable laminate and the complexities of the shell governing equations. Numerical methods are, therefore, the only practically viable option for computing the dependent field variables and the stability characteristics. In the context of constant-stiffness curved panels, previous authors have mostly applied incremental procedures to the post-buckling problem, such as the Newton–Raphson or Riks algorithms (Panda and Ramachandra, 2010; Thornburgh and Hilburger, 2006). Such methods are, of course, useful when a ‘deep’ non-linear solution is required, they do however, produce large quantities of data, and have relatively large computational requirements: two characteristics which are not well suited to optimisation problems. Asymptotic methods, on the other hand, have the advantage that they may project an approximation of the non-linear solution from a point of interest (i.e. a bifurcation point) and hence only require a small number of variables. They are useful, therefore, for estimating the behaviour of a system without actually following an equilibrium path. Previous implementations of asymptotic methods for shell problems have mostly been produced within the finite-element framework and for general structures (Casciaro et al., 1998; Garcea et al., 2009; Kheyrkhahan and Peek, 1999).

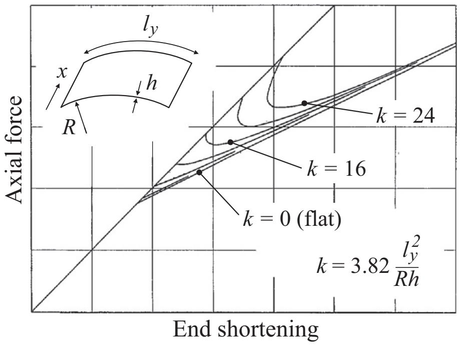

Fig. 1. Load vs. end-shortening curves for isotropic ($\nu = 0.3$) curved panels, under compression in the axial (x) direction (with a range of initial curvatures k). Radial displacements, tangential rotations and axial rotations are zero on all edges. Tangential and axial displacements are free on all edges. [Reproduced from Pope (1965)].