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Generalized shear of a soft rectangular block

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ABSTRACT

The problem of the simple shear of a block has been treated in terms of a shear displacement, applied uniformly in a lateral direction and assumed to be a linear function of the height above the base. In this paper, simple shear is generalized: the shear displacement is neither uniform in the lateral direction nor necessarily a linear function of the height. Using second-order isotropic elasticity, the analytical solutions show that the shear displacements are characterized by the product of sine and hyperbolic sine functions of the height and depth variables, respectively. The height dependence of the shear displacement is predicted to be a combination of linear and sinusoidal functions, and is verified against the test data of agar–gelatin cuboidal blocks. If the gravity effect is incorporated, a quadratic dependence on height is additionally predicted. The calculation of stresses reveals the presence of not only negative normal stresses but also sinusoidally varying shear stresses on the lateral planes tending to distort the block about the height direction. These results can be of great importance in tissue/cell mechanics.

1. Introduction

In nonlinear elasticity, the existence of normal stresses under simple shear has long been recognized (Rivlin, 1948) and so has the lengthening of a cylinder under pure torsion (Poynting, 1909). There has been a recent interest in the shearing and torsion of soft materials such as biopolymers and hydrogels, e.g., Storm et al. (2005), Janmey et al. (2007), Kang et al. (2009), Wu and Kirchner (2010), Destrade and Saccomandi (2010), Destrade et al. (2012), Mihai and Goriely (2011, 2013), and Horgan and Murphy (2011). A significant finding of some of these works is the existence of negative normal stresses or the negative Poynting effect when certain soft solids are subjected to shear or torsion. Negative normal stresses occur if the sheared faces of a block are drawn together under shear, and equivalently the negative Poynting effect occurs if a cylindrical rod shortens under torsion. The shearing of soft biological materials is common in their physiological environments (Horgan and Murphy, 2011), and understanding of nonlinear phenomena such as the Poynting effect is of great importance. For instance, the large stresses generated by the Poynting effect can result in a significant effect on the overall force balance in the cytoskeleton under shear (Janmey et al., 2007), and the Poynting effect is important in understanding the interaction between surgical instruments and tissues (Misra et al., 2010). In this paper, the simple shear problem will be generalized and investigated in some detail.

In the strain formulation of the simple shear problem, e.g., Rivlin (1948), the starting point is the deformation of a block as given by:

$$x = X + \kappa Y, \quad y = Y, \quad z = Z \tag{1.1}$$

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where (X, Y, Z) and (x, y, z) are the referential and current Cartesian coordinates respectively, and κ is the shear deformation. Murnaghan (1951) also started from Eq. (1.1) in studying simple shear within the framework of second-order elasticity. Calculation of the stresses then reveals that normal stresses must exist to support such a deformation. Recently, Destrade et al. (2012) pursued a stress formulation for hyperelastic materials in which the starting point is a uniform shear stress $\sigma_{12}=\sigma_{21}$. The resulting deformation is given by:

$$x=\lambda_1 X+\lambda_2\sqrt{1-{\lambda_1}^2{\lambda_2}^{-2}}Y, \quad y=\lambda_2 Y, \quad z=\lambda_3 Z, \tag{1.2}$$

where $\lambda_1, \lambda_2,$ and λ_3 are the principal stretches. This means that under a pure shear stress the deformation consists of a simple shear of the type given by Eq. (1.1) and triaxial extensions. Hence, under the strain formulation the resulting stress field consists of shear stresses and triaxial normal stresses, while under the stress formulation the resulting deformation field consists of simple shear and triaxial stretches. Mihai and Goriely (2013) considered generalized shear in which Eq. (1.1) is modified into:

$$x=X+\kappa(Y), \quad y=Y, \quad z=Z, \tag{1.3}$$

where κ is now a function of Y. Specifically, $\kappa(Y)$ is assumed to be of the form of $Y^2/2$, i.e., a quadratic function of Y. Destrade and Saccomandi (2010) also considered the effect of gravity in the form of a body force $-\rho_0 g$ (ρ_0 is the mass density and g is the gravitational constant). They took the deformation to be:

$$x=X+u(Y), \quad y=Y+v(Y), \quad z=Z, \tag{1.4}$$

where, upon solving the governing equations and assuming the Murnaghan strain energy density, $u(Y)$ was found to be the sum of linear and quadratic parts.

In this paper, we consider the generalized shear of a block with the X-axis as the shearing direction, the Y-axis the vertical direction and the Z-axis the depth direction. The deformation is taken to be

$$x=X+u(Y,Z), \quad y=Y+v(Y), \quad z=Z, \tag{1.5}$$

where the specific form for $u(Y, Z)$, under various prescribed displacement profiles at the top of the block, is solved from the governing equilibrium equations. Two new features of the formulation are noted: $u(Y, Z)$ is not assumed *a priori* and it could be a function of Z. The prescribed displacements considered are constant, quadratic and sinusoidal functions of Z. The dependence of u on Y, i.e., the deformed block shows a curved rather than straight lateral profile, has been noted by Gardiner and Weiss (2001) during their finite element simulation of the simple shear of soft tissues. The form of this dependence has been assumed to be quadratic in Eq. (1.3), and predicted to be the sum of linear and quadratic terms in the gravity formulation of Eq. (1.4). In our work, we will show that, in the absence of gravity, u is the sum of a linear function of Y and a function of the product $\sin Y \times \sinh Z$, even if the prescribed displacement at the top of the block is uniform, i.e., $u(Y=L_Y, Z)=\text{constant}$, where L_Y is the height of the block. Hence, a block of soft material subjected to a constant shear displacement at the top surface is curved in both the X–Y and X–Z planes (but not in the Y–Z plane).

We also use the Murnaghan strain energy density as the constitutive law. The gravity effect can also be taken into account as was done previously by Destrade and Saccomandi (2010). We compare the shape of deformed blocks of agar–gelatin with the theoretical predictions. Furthermore, we explore the dependence of negative or positive normal stresses on the elastic constants. The paper is organized into five sections: introduction, formulation, numerical results, discussion and conclusions.

2. Second-order formulation of the generalized shear of a rectangular elastic block

2.1. Definition of problem

We consider a homogeneous isotropic rectangular block of dimensions L_X, L_Y and L_Z under generalized shear and gravity as shown in Fig. 1(a). In reference Cartesian coordinates (X, Y, Z), the shear displacement is assumed to be $(U, V, 0)$, so that the final coordinates (x, y, z) are given by $(X+U, Y+V, Z)$. Here $U=u_1(Y,Z)+ku_2(Y,Z)$denotes the shear displacement in the X-direction, with u_1 and u_2 representing the linear and nonlinear components, respectively. U is taken to be a function of both Y and Z, allowing for the possibility of a non-uniform (generalized) shear which might exist in a complex setting such as a physiological environment. Similarly,$V=v_1(Y)+kv_2(Y)$ denotes the displacement in the Y-direction resulting from gravity, with v_1 and v_2 denoting the linear and nonlinear components. However, V is solely a function of Y as this is the direction of gravity. If gravity is neglected, $V=0$. Note also that k is a marker indicating the order of approximation of the theory. In the subsequent formulation, we retain terms up to k^2 for second-order theory.

Fig. 1(b) shows the possible deformed configuration of the block, which is fixed to the bottom plane $Y=0$, and subjected to prescribed displacement at the top face $Y=L_Y$. The linear part of the prescribed displacement is proportional to Y, and the nonlinear part is described by any g(Z) which is symmetric with respect to $Z=L_Z/2$. Furthermore, we prescribe a moment M_{xZ} about the Z-axis, which is associated with an induced out-of-plane shear.