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On the static and dynamic analysis of inflated hyperelastic circular membranes

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A B S T R A C T

This paper considers certain aspects of static and dynamic analysis of inflated unstretched and prestretched flat circular hyperelastic membranes. The problem is both geometrically and materially nonlinear. The governing equations of equilibrium, and the equations of small amplitude dynamics are obtained using the variational formulation. The equilibrium configuration of the membrane is obtained by solving a two-point boundary value problem exploiting a scaling symmetry of the equations of equilibrium. Interestingly, in certain cases, beyond a certain inflation of the membrane, the Gaussian curvature flips sign (positive to negative) near the periphery of the membrane leading to neck formation and impending wrinkling condition. The dynamics of perturbations over the static shape have been studied considering both constant pressure and adiabatic conditions. Two remarkable new phenomena, namely a symmetry breaking torsional mode instability via a supercritical pitchfork bifurcation, and a stretch induced softening behaviour of the membrane, are revealed through the analysis.

1. Introduction

Inflatable membrane structures have been the subject of interest in recent years due to the unique features they offer. They are of much lower weight and cost, can be packed compactly and deployed quickly. Hence, they are ideally suited for applications in terrestrial and space structures (Jenkins, 2001). Some other common applications include scientific ballooning, shock and vibration absorbers, thermal shields, and bio-engineering and medical devices. Inflated membranes exhibit a number of interesting and counterintuitive phenomena on account of material and geometric nonlinearities and their interaction. The dynamics and stability of such structures are of importance in a number of applications.

The pioneering work of Mooney (1940) on nonlinear elasticity forms the basis for the analysis of large deformations of structures. Further developments in this field are based on the works by Green and Adkins (1960), Ogden (1984) and Taber (2004). Membranes are usually made of long-chain polymeric materials which exhibit very complex mechanical behaviour. They are treated as nonlinear hyperelastic materials. The commonly used constitutive models are the neo-Hookean and Mooney–Rivlin models (see, e.g., Treloar, 1975). Apart from these, there are other constitutive models proposed by Ogden (1984), Arruda and Boyce (1993), Gent (1996), etc.

Inflatable structures undergo large deformation and exhibit geometric nonlinearity in the stretch displacement relations. Inflation problems are, in general, difficult to analyze except in some special geometries with symmetry (see, e.g., Corneliussen and Shield, 1961; Foster, 1967; Hart-Smith and Crisp, 1967). Yang and Feng (1970) have analyzed the problem of nonlinear axisymmetric deformation of membranes under different loading conditions using the Newtonian formulation.

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Recently, Patil and DasGupta (2013) have proposed a technique to solve the two-point boundary value problem associated with the axisymmetric inflation problem of circular flat membranes exploiting the scaling invariance property of the equations of equilibrium. Tamadapu and DasGupta (2013) have studied finite deformations of a hyperelastic toroidal membrane of initially circular cross-section. The problem is formulated using the variational approach, and the stability of the inflated configurations of the membrane in terms of impending wrinkling has been investigated.

Under large deformations, the material and geometric nonlinearities interact to exhibit certain curious instabilities and bifurcations. The tensile instability phenomenon and bifurcation to non-spherical modes, observed in spherical balloons, has been investigated using perturbations over a deformed state by Feodos'ev (1968), and verified later along with experiments by Alexander (1971). Axisymmetric necking in pressurized spherical membranes has been addressed by Needleman (1976). Several features associated with the axisymmetric non-spherical modes of deformation of an elastic spherical membrane with imperfection under internal pressure have been studied by Needleman (1977). The axisymmetric equilibrium states have been determined using the Ritz–Galerkin method. The occurrence of limit point instability, which is usually attributed to the material nonlinearity, is a well-known phenomenon in hyperelastic membranes (see, e.g., Patil and DasGupta, 2013). It has been recently shown by Tamadapu et al. (2013) that the limit point instability pressure has an interesting geometry and material dependent scaling behaviour involving certain universal constants related to the geometry.

Studies on dynamic inflation of a spherical membrane around a static fixed point have been reported by Verron et al. (1999). Various inflation regimes based on the material parameter and pressure have been investigated. The nonlinear vibrations of radially stretched circular membrane subjected to finite deformation due to uniform forcing at its edge have been presented by Goncalves et al. (2009). They have studied the nonlinear frequency–amplitude relation along with the different bifurcations. Tamadapu and DasGupta (2012) have studied the in-plane surface modes of a toroidal membrane considering isotropic and anisotropic homogeneous material properties. The effect of curvature on the dynamics has been clearly brought out. Further the in-plane dynamics of two-dimensional homogeneous and isotropic elastic membranes of constant curvature has been discussed by DasGupta and Tamadapu (2013).

In the existing literature, the dynamics of uninflated membranes with different geometric and material parameters has been studied. However, the static and dynamic aspects of the inflated membrane structures have not been explored completely. The study of dynamics of inflated membranes is expected to clarify certain stability issues, bifurcations, necking, etc. The effect of geometric and material parameters on the dynamics and stability of membrane structures can present interesting aspects. These observations motivate the present study.

In this work, we have considered the static and dynamic analysis of inflated circular hyperelastic membranes. In the variational formulation, we have considered the Mooney–Rivlin strain energy density function, along with the internal energy of the inflating gas, and kinetic energy of the membrane. The equilibrium configuration is perturbed along the radial, circumferential and transverse directions to obtain a small amplitude general perturbation dynamics. The dynamic analysis has been carried out under constant pressure, and adiabatic conditions. The static inflation problem is obtained as a special case of this formulation where the time dependent perturbations are dropped out. Beyond a certain inflation, a neck with negative Gaussian curvature is formed at the periphery of the membrane. This eventually leads to an impending wrinkling state in the circumferential direction, which destabilizes the membrane. The effects of material parameter and prestretch on the wrinkling instability are studied. The variations in the eigenfrequency spectrum of the perturbation dynamics with the inflation pressure, strain hardening parameter and prestretch are determined. A counter-intuitive stretch induced softening behaviour is found in the membrane. This corroborates a similar conclusion reported recently by Patil and DasGupta (2013). A new type of symmetry breaking torsional mode instability via a supercritical pitchfork bifurcation is observed. This instability brings out a new necking mechanism in the membrane which breaks the axisymmetry of the structure.

2. Kinematics of deformation

Consider an initially flat circular unstretched membrane, which is stretched radially and then inflated under constant pressure as shown in Fig. 1. From an initial position $P_1(r,\theta,0)$ (State 1), a material point moves, upon radial stretching, to $P_2(\overline{\rho},\theta,0)$ (State 2), and then moves further, upon inflation, to $P_3\,(\rho_0,\theta,\eta_0)$ (State 3), as shown in the figure. Here, we assume $\overline{\rho}(r)=rR_f/R_0$, and the static axisymmetric radial and transverse deflections of the inflated membrane are represented as, respectively, $\rho_0(r)$ and $\eta_0(r)$ in the cylindrical coordinate system.

Let the inflated shape be perturbed spatio-temporally about the equilibrium configuration as

$$\rho(r,\theta,t)=\rho_0(r)+\varepsilon u(r,\theta,t),\quad \beta(r,\theta,t)=\theta+\varepsilon v(r,\theta,t),\quad \eta(r,\theta,t)=\eta_0(r)+\varepsilon w(r,\theta,t) \tag{1}$$

where ε is a book-keeping parameter, and u, v and w represent the perturbation field, as shown in Fig. 1. The Cartesian coordinates of a point in the perturbed configuration may be represented as

$$z_1=\rho(r,\theta,t)\cos\beta(r,\theta,t),\quad z_2=\rho(r,\theta,t)\sin\beta(r,\theta,t),\quad z_3=\eta(r,\theta,t). \tag{2}$$

The line element (infinitesimal distance measure) on the membrane in the undeformed and perturbed configurations is given by, respectively,

$$\mathrm{d}s^2=\mathrm{d}r^2+r^2\,\mathrm{d}\theta^2+\mathrm{d}\xi^2=g_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j \text{ (summation convention)}$$
$$\mathrm{d}S^2=(\rho_{,r}^2+\rho^2\beta_{,r}^2+\eta_{,r}^2)\,\mathrm{d}r^2+(\rho_{,\theta}^2+\rho^2\beta_{,\theta}^2+\eta_{,\theta}^2)\,\mathrm{d}\theta^2+\Lambda_3^2\,\mathrm{d}\xi^2=G_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j$$