Journal of the Mechanics and Physics of Solids 64 (2014) 468–482

Contents lists available at ScienceDirect

Journal of the Mechanics and Physics of Solids

journal homepage: www.elsevier.com/locate/jmps

Annular inhomogeneities with eigenstrain and interphase modeling

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ARTICLE INFO

Article history:
Received 12 August 2013
Received in revised form
21 November 2013
Accepted 4 December 2013
Available online 18 December 2013

Keywords:
Inhomogeneity
Eigenstrain
Interphase
Surface tension
Driving force

ABSTRACT

Two and three-dimensional analytical solutions for an inhomogeneity annulus/ring (of arbitrary thickness) with eigenstrain are presented. The stresses in the core may become tensile (for dilatational eigenstrain in the annulus) depending on the relative shear moduli. For shear eigenstrain, an “interface rotation” and rotation jumps at the interphase also occur, consistent with the Frank–Bilby interface model. A Taylor series expansion for small thickness of the annulus is obtained to the second-order as to model thin interphases, with the limit agreeing with the Gurtin–Murdoch surface membrane, but also accounting for curvature effects.. The Eshelby “driving forces” on a boundary with eigenstrain are calculated, and for small, but finite, interphase thicknesses they account for the interaction of the two interfaces of the layer, and the next order term may induce instabilities, for some bimaterial combinations, if it becomes large enough to render the driving force zero. It is also proven that for 2-D inhomogeneities with eigenstrain the stresses have reduced material dependence for any geometry of the inhomogeneity. The case when the outer boundary of the inhomogeneity annulus with eigenstrain is a free surface is also analyzed and agrees with classical surface tension results in the limit, but, moreover, the thick free surface terms (next order in the expansion depending on the radius) are also obtained and may induce instabilities depending on the bimaterial combinations. Applications of inhomogeneity annuluses with eigenstrain are wide and include interphases in thermal barrier coatings and coated particles in electrically/thermally conductive adhesives.

1. Introduction

The Eshelby (1957) ellipsoidal inclusion solution and the Eshelby equivalent inclusion method for inhomogeneities have had widespread applications to composite materials over the last 50 years. Here we present solutions and properties of inhomogeneities with transformation strain, or eigenstrain (a term introduced by Dundurs, 1967), when the inhomogeneity is an annulus embedded in a matrix, which cannot be solved by the Eshelby (1957, 1961) equivalent inclusion method through an algebraic system of linear equations. The inhomogeneity annulus cannot be solved by superposition of Eshelby inclusions, since the Eshelby equivalent inclusion method would require to have a position dependent loading (of the outer field of the inner inclusion) in the equivalent eigenstrain system of equations, rather than the constant coefficients that are due to the Eshelby property for the ellipsoid. For an updated treatise on inclusions and inhomogeneities with eigenstrain

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0022-5096/$ - see front matter
http://dx.doi.org/10.1016/j.jmps.2013.12.003

and applications we refer to Qu and Cherkaoui (2006). Attention is drawn to a related very recent paper on spherical multi-inhomogeneous inclusions by Shodja and Khorshidi (2013), with which we strongly disagree. To quote them, at the end of Section 5, " *Remark 4: Therefore, interestingly, the uniform eigenstrain fields … give rise to a center of dilatation at* $r=0$*. Non-uniform eigenstrain fields produce stress and strain fields … with logarithmic singularities at* $r=0$ *as well.*" By the very principles of analytic function theory, finite eigenstrain *cannot* induce stresses and strain fields with singularities, except at points of geometric discontinuities, such as corners of inclusions (Rodin, 1996), or corners at strips of eigenstrain meeting a free surface (Dundurs and Markenscoff, 2009). These singularities have been routinely excluded a priori in all the literature (e.g. Duan et al., 2005 for the coated inclusion problem, Duan et al., 2008; Hashin, 2002). The current paper contains the correct solution without singularities.

The applications of annular inhomogeneities with eigenstrain range from modeling interphases at grain boundaries (e.g. Hirth et al., 2012) to thermal barrier coatings (Pollock et al., 2012), and to coated particles in thermally/electrically conductive adhesives (e.g. Morris and Liu, 2007). The literature is vast on coated inhomogeneities and interphase layers between the inclusions and the matrix, or surface layers (but no eigenstrain loading in the inhomogeneous annulus has been previously considered), and we will selectively reference Steigmann and Ogden (1997, 1999), Sharma and Ganti (2004), Duan et al. (2005, 2008), Hashin (2002), Dingerville et al. (2005), Dingreville and Qu (2008), Benveniste (2013), and Mohammadi et al. (2013), while a more comprehensive list of references on the material aspects of interphases is provided in the long article of Hirth et al. (2012).

Interesting features of the analytical solutions presented here are, that, for dilatational eigenstrain in the inhomogeneous ring/annulus, the stress in the core actually depends on the difference $(\mu_1-\mu_2)$ of the shear moduli of the matrix and the annulus, both in 3-D (called "annulus") and in 2-D ("ring"), and, thus, the inhomogeneous annulus induces hydrostatic tension inside the core when the annulus is stiffer in shear than the matrix. Thus, if the interphase thickness is appreciable with respect to the core radius, the tensile stress in the core induced by the interphase may be significant and have implications for nanomaterials. This inhomogeneity solution elucidates further the Eshelby inclusion result for dilatational eigenstrain that the value of the bulk modulus of the matrix is "irrelevant" (Eshelby, 1961) (and only the shear modulus matters).

The annular inhomogeneities with eigenstrain may be useful to model interphases of finite thickness. For small inhomogeneity thicknesses h, the stresses and rotation inside the interphase are also obtained by a Taylor series expansion in the thickness parameter, and, as the thickness h goes to zero, the stresses are found to depend only on the elastic constants of the annulus/ring, which is consistent with the membrane interface (or free surface) models (such as Gurtin and Murdoch, 1975). However, the presence of shear eigenstrain in a layer of vanishingly small thickness also produces a nonzero "interface rotation", not obtainable otherwise. The interface rotation is not a feature of other surface stress models (such as the one of Gurtin and Murdoch, 1975), but has been experimentally observed, and is a feature of the Frank–Bilby interface model in the recent seminal paper of (Hirth et al., 2012).

The eigenstrain in the annulus gives rise to *"driving forces"* (Eshelby forces) on the interfaces between the matrix and the interphase, which are computed according to $f=-<\sigma_{ij}>[[\varepsilon^*_{ij}]]$ (Eshelby, 1977), and same in dynamics (Markenscoff and Ni, 2010), on each interface of the *finite* interphase layer thickness, accounting also for the interaction with the other interface of the layer. As the thickness tends to zero, the hoop stresses and the driving force depend on the layer material constants *only*, but, for small *finite* thickness h the driving force has the next order h/a positive term which is proportional to the $(\mu_1-\mu_2)$ difference of the matrix/interphase shear moduli and the bimaterial Dundurs constants α, β. This term may induce *instabilities* of the interface if it becomes large enough to be of the order of the leading term due to hoop tension so that the driving force becomes zero.

We also present solutions for a thin spherical inhomogeneous annulus with eigenstrain, where, again, the hoop stresses depend only on the material constants of the annulus. The leading term of the driving force at the interface is found to be equal in value to the driving force for a spherical inclusion of material 2 (in a matrix of material 2) with dilatational eigenstrain in the core, as derived by Gavazza (1977), Eshelby (1977), and Markenscoff and Ni (2010). This should be expected physically, since the thin annulus stress field "knows" only the material of the annulus and the eigenstrain in it (there is no other characteristic length), and the Gavazza–Eshelby's result is valid for inclusions of any shape (Eshelby, 1957, 1961). Moreover, the presence of eigenstrain gives rise to a *driving force*, which controls the *stability* of the interface. The obtained finite interphase solution for the driving force also accounts for the *interaction* of the driving force on one interface of the layer with the other interface. In addition, the "excess energy" (Dingreville and Qu, 2008) inside the interphase, here associated with the eigenstrain, can be calculated according to Eshelby (1961).

We also present here the solutions for a finite radius ring/annulus of eigenstrain enclosing a matrix and being free of traction on the outside. The generated stresses are hoop stresses of "surface tension", which, for vanishingly small thickness, again tend to a constant depending only on the elastic moduli of the thin shell. The driving force on both the free surface and the interface to the leading order depends on the surface moduli and the eigenstrain, but the next order h/a (where a is the core radius) term may become large (for some bimaterial combination differences) and be of the order of the surface tension term, in which case the free surface and matrix/free surface interfaces may become unstable.

In the obtained solutions for 2-D ring inhomogeneities with eigenstrain, the stresses are expressed in terms of the Dundurs bimaterial constants $(\alpha-\beta)$ with a coefficient E^*, as common multiplier in all the fields. It is shown here that, by applying the CLM theorem (Cherkaev et al., 1992), this coefficient E^* has a reduced dependence on the elastic constants. In addition, we show that the reduced material constants dependence of the stresses is valid for *arbitrary* shapes of