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On a general numerical scheme for the fractional plastic flow rule

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A B S T R A C T

This paper presents a general numerical scheme for the fractional plastic flow rule, dedicated to a wide class of materials manifesting the non-normality of plastic flow and induced plastic anisotropy. To determine the vector of the plastic flow, a special numerical procedure has been developed, which is applicable for any smooth and convex yield function. The obtained approximation is verified based on an analytical solution. The paper also presents a set of numerical results for the generalised Drucker–Prager model.

1. Introduction

Natural and man-made materials are very often forced to operate in both the reversible (elastic) and irreversible (plastic) ranges. Optimal utilization of these ranges requires a detailed (numerical) analysis under the prescribed mechanical conditions, which depend on the application. One of the challenging tasks herein is to reproduce by the constitutive model, in the irreversible range, the phenomena of plastic anisotropy and/or effects causing non-associative flow resulting in plastic volume change.

The group of materials which manifest plastic anisotropy and non-associative flow is very broad including, e.g.: (i) geomaterials: rocks (Maier and Hueckel, 1979; Lubarda et al., 1996), granular materials (Drescher and Detournay, 1993; Wang et al., 2001), clays (Ling et al., 2002; Jiang et al., 2012), and reinforced soils (Michalowski and Zhao, 1995); (ii) concrete (Palaniswamy and Shah, 1974; Hu and Schnobrich, 1989); (iii) ceramics (Reyes-Morel and Chen, 1988; Radi and Bigoni, 1993); (vi) composites (Lei and Lissenden, 2007); and (v) metallic materials (McDowell, 2008; Taherizadeh et al., 2011). Nevertheless, it should be emphasised that the importance of both these phenomena depends on the specific material and the applied mechanical conditions under which it operates. Moreover, although often modelled using the same mathematical formula, the physical meaning of plastic anisotropy and non-associativity of plastic flow differs depending on the material. As an example, for geomaterials it appears as a result of the dilatancy effect, which accompanies the shearing process (Reynolds, 1885; Michałowski and Mróz, 1978); in metallic materials it results from the dislocation nucleation (Ziegler, 1983; Dao and Asaro, 1993; Racherla and Bassani, 2007), nucleation of voids during plastic flow (Marin and McDowell, 1998), or the breakup of grains into disoriented, blocky subgrains (Hughes et al., 1997; Steinmann et al., 1998).

With respect to the mathematical modelling of irreversible mechanical processes, recently, emphasis has been placed on the application of the *fractional calculus* (FC), the branch of mathematical analysis which deals with differential equations of an arbitrary order (Nishimoto, 1984–1991; Podlubny, 1999; Kilbas et al., 2006). Herein, the fundamental aspect is that the fractional derivative introduces the non-local effects by definition, and with a limited number of additional parameters, a family of models well suited to mimic the behaviour of a wide class of materials is obtained. One can even mention here the statement by Katsikadelis (Katsikadelis, 2015) regarding the meaningful role of FC in modern mechanics: *"It would not be excessive to say that simulating physical systems using only integer-order derivatives is similar to doing arithmetic (algebra) using only integer numbers"*.

It should be pointed out that one can obtain different generalisations of classical plasticity models, depending on which variable the fractional operator acts upon. Let us mention in this respect the papers: (J. et al., 2012) where a 1D non-local in strain state formulation of plasticity was proposed; (Sumelka, 2014b; Sumelka and Nowak, 2016) where a 3D non-local in the stress state model was defined; (Sumelka, 2014a) where a non-local in space plasticity model was considered; and (Suzuki et al., 2016) where a 3D non-local in time model of plasticity was shown. It is important, that these different models can also be coupled depending on the need; however the common problem regarding FC will remains,

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namely that the analytical solution is rarely possible, and at once the numerical approximations are needed.

This paper is devoted to the above mentioned aspect of looking for approximated solutions for plasticity models generalised using FC, concentrating on the fractional flow rule concept formulated in Sumelka (2014b) and developed in Sumelka and Nowak (2016). The numerical scheme proposed in this paper extends previous results presented in Sumelka and Nowak (2016) and applies to any smooth and convex yield surface. To obtain this flow chart, the approximation (among other known concepts (Gorenflo et al., 2002; Sun and Wu, 2006; Lin and Xu, 2007)) of the fractional operator proposed in Odibat (2006) has been selected. As in Sumelka and Nowak (2016), the final verification analysis concentrates on induced plastic anisotropy and volume change effects in the plastic range due to the non-associativity of the fractional flow. The algorithm has been implemented in the Abaqus/Explicit code and 3D simulations of uniform and non-uniform tension of a cylindrical specimen have been performed.

The paper is structured as follows. In Section 2, the concept of fractional flow rule is discussed. Section 3 deals with the general computational scheme for the fractional plastic flow for smooth and convex yield surface. Section 4 is devoted to the verification of the proposed algorithm. Section 5 concludes the paper.

2. The fractional flow rule concept

2.1. Remarks on fractional calculus

Fractional Calculus (FC) deals with the integrals and derivatives of arbitrary (even complex) orders as mentioned (Nishimoto, 1984–1991; Podlubny, 1999; Kilbas et al., 2006). Unlike the classical integer order integration and differentiation, where a single definition for both operations exists, FC introduces an infinite number of possible definitions of their generalisations (Oliveira and Machado, 2014). Hence, the specific type of fractional derivative applied in the mathematical model (in a phenomenological sense) should be argued as the one that gives the best mapping of experimental observations.

In this paper, the both sided fractional derivative obtained as a linear combination of the left-sided and the right-sided Caputo-type derivatives will be applied. We call such an operator a Riesz–Caputo (RC) type operator (Agrawal, 2007; Frederico and Torres, 2010) and we define it as follows.

The generalized fractional integral K_P^α of a function f with argument t is defined as (Odzijewicz et al., 2013)

$$(K_P^\alpha f)(t) := p\int_a^t k_\alpha(t,\tau)f(\tau)d\tau + q\int_t^b k_\alpha(\tau,t)f(\tau)d\tau, \tag{1}$$

where $P=\langle a,t,b,p,q\rangle$ is the parameter set, $t\in[a,\, b]$, p, q are real numbers, $k_\alpha(t,\ \tau)$ is a kernel which may depend on α, and f is any function defined almost everywhere on $(a,\, b)$ with values in $\mathbb{R}$. If k_α is a difference kernel, i.e. $k_\alpha(t,\tau)=k_\alpha(t-\tau)$ and $k_\alpha\in L_1([0,b-a])$ then $K_P^\alpha : L_1([b,a])\to L_1([b,a])$ is well defined, bounded and linear. Special cases of the operator K_P^α are obtained for the kernel ($\alpha>0$)

$$k_\alpha(t-\tau) = \frac{1}{\Gamma(\alpha)}(t-\tau)^{\alpha-1}, \tag{2}$$

namely, if $P=\langle a,t,b,1,0\rangle$, we have

$$(K_P^\alpha f)(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-\tau)^{\alpha-1}f(\tau)d\tau =: ({}_aI_t^\alpha f)(t), \tag{3}$$

or, if $P=\langle a,t,b,0,1\rangle$, we obtain

$$(K_P^\alpha f)(t) = \frac{1}{\Gamma(\alpha)}\int_t^b (\tau-t)^{\alpha-1}f(\tau)d\tau =: ({}_tI_b^\alpha f)(t), \tag{4}$$

where Γ is the Euler gamma function. The integrals $({}_aI_t^\alpha f)(t)$ and $({}_tI_b^\alpha f)(t)$ are commonly known as the left and right Riemann–Liouville fractional integrals, respectively.

The left-sided and the right-sided Caputo derivatives are then defined as:

$$(B_P^\alpha)f(t) := {}_a^CD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)}\int_a^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}}d\tau, \quad \text{for} \quad t>a, \tag{5}$$

and

$$-(B_P^\alpha)f(t) := {}_t^CD_b^\alpha f(t) = \frac{(-1)^n}{\Gamma(n-\alpha)}\int_t^b \frac{f^{(n)}(\tau)}{(\tau-t)^{\alpha-n+1}}d\tau, \quad \text{for} \quad t<b, \tag{6}$$

where $n=\lfloor\alpha\rfloor+1$, $\lfloor\cdot\rfloor$ denotes the floor function and $(B_P^\alpha)f(t) := K_P^{n-\alpha}\circ\frac{d^n}{dt^n}$.

Finally, the RC type fractional derivative is

$$D^{\,\alpha} f(t) = {}^{RC}_{\ a}D_b^\alpha f(t) = \frac{1}{2}\left({}_a^CD_t^\alpha f(t) + (-1)^n\, {}_t^CD_b^\alpha f(t)\right). \tag{7}$$

It should be pointed out that for the integer values of α, Eq. (7) reduces to the classical integer order derivative, and independently of α RC derivative of a constant gives zero.

2.2. The plasticity model with the fractional plastic flow rule

The concept of fractional plastic flow was presented in papers (Sumelka, 2014b, 2014c; Sumelka and Nowak, 2016). This new idea is based on the standard assumptions of the theory of plasticity, i.e.

$$\dot{\varepsilon} = \dot{\varepsilon}^e + \dot{\varepsilon}^p, \tag{8}$$

$$\dot{\sigma}^e = \mathcal{L}^e : \dot{\varepsilon}^e, \tag{9}$$

$$\dot{\varepsilon}^p = \Lambda\mathbf{p}, \tag{10}$$

but the direction of flow $\mathbf{p}$ is computed using a fractional derivative, namely

$$\mathbf{p} = \underset{\sigma}{D}^{\,\alpha} F, \tag{11}$$

where $\boldsymbol{\varepsilon}$ stands for the total second order strain tensor, $\boldsymbol{\varepsilon}^e$ and $\boldsymbol{\varepsilon}^p$ denote elastic and plastic strains, $\boldsymbol{\sigma}^e$ denotes the second order Cauchy stress tensor, $\mathcal{L}^e$ denotes the fourth order elastic constitutive tensor, Λ is a scalar multiplier, F is the yield function, D_σ^α denotes partial fractional differentiation of RC type, and α denotes the order of derivative (the order of flow). Eq. (11) is called the *fractional flow rule*, which makes the direction of $\mathbf{p}$ dependent on some virtual neighbourhood of the stress state at the material point (defined by terminals in the RC operator - cf. Fig. 1 and detailed discussion in Sumelka and Nowak (2016)). For $\alpha=1$, a smooth passage to the classical associated plastic flow is obtained.

It is easy to understand, looking at Eqs. (5) and (6), that the analytical solution for $\mathbf{p}$ does not exist in general, even for a very basic function used to define F. Therefore, in the next section a numerical approximation of $\mathbf{p}$ assuming a smooth and convex yield function is given and verified subsequently.

2.3. Remarks on thermodynamic restrictions

The presented plasticity model, including the fractional flow rule, belongs to the class of phenomenological models with internal state variables. The applicability of the model, in the sense of a range of physically allowable crucial model parameters (i.e. terminals and order of flow - cf. Eq. (11), results from the thermodynamic restrictions, and depends on the type of yield function considered. Constraints to be fulfilled are obtained in the classical manner, as presented below.