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Uniform hydrostatic thermal stress fields inside two interacting inclusions of irregular shape and related problem

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ABSTRACT

By using the technique of conformal mapping, we construct the irregular shapes of two interacting elastic inclusions with internal uniform hydrostatic thermal stresses due to a uniform change in temperature. It is found that the hoop stresses along the two inclusion/matrix interfaces on the matrix side are constant. The internal uniform hydrostatic thermal stresses within the two inclusions and the hoop stresses along the two interfaces are independent of the specific inclusion shapes. Numerical results are presented to validate and demonstrate the findings. Furthermore, we design the optimum shapes of the two inclusions that permit internal uniform stresses when uniform anti-plane eigenstrains are imposed on the two inclusions of irregular shape.

1. Introduction

Since the pioneering work on a single ellipsoidal inclusion in an infinite space by Eshelby (1957), extensive researches have been conducted in this area (see e.g. Luo and Weng, 1987; Gao, 1991; Nemat-Nasser et al., 1993; Dong et al., 2002; Liu et al., 2004, 2006; Fang and Liu, 2006; Jin and Fang, 2008; Fang et al., 2009, 2010; Hashemi et al., 2010; Hoh et al., 2010; Wang, 2010; Feng et al., 2011; Zhou et al., 2011, 2012; Markenscoff, 2010, 2012; Wang and Zhou, 2012; Zhou, 2012). Many other works on inclusions can be found in a recent review by Zhou et al. (2013).

The study of inclusions has been complicated and challenging when two or more inclusions are present because their interactions should be taken into account, although many studies have neglected the interactions to certain extent for simplicity. However, such study is badly needed due to the fact that composites are materials that contain multiple inclusions.

The "constant strength" design requirement was first proposed by Cherepanov (1974). In this requirement, the hoop stress (or the tangential normal stress) is constant everywhere on the boundary of a hole or along the interface between the elastic inclusion and surrounding material (Cherepanov, 1974; Wheeler, 1992; Bjorkman and Richards, 1976; Ru, 1999a; Wang, 2012). Most of the investigations on composite structures with "constant strength" are confined to the case of mechanical loadings. One aim of this research is to design fibrous composites with "constant strength" due to the thermal loading of a uniform change in temperature.

On the other hand, the internal uniform hydrostatic stress state within an elastic inclusion is preferred because this special kind of internal stress state will produce a constant interfacial stress distribution, thus eliminating any stress peak along the interface (Ru, 1999a).

Under a uniform temperature change, an isolated circular inclusion is optimum in the sense that the induced internal thermal stress field is uniform and hydrostatic, and meanwhile the hoop stress on the matrix side is constant along the entire circular interface (Muskhelishvili, 1953). In this work, by means of the conformal mapping technique, we have endeavored to design the optimum irregular shapes of two interacting elastic inclusions that permit internal uniform hydrostatic stress fields under a

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uniform temperature change. It is found that the obtained composite structure also satisfies the "constant strength" design requirement proposed by Cherepanov (1974). Furthermore, we have solved a related anti-plane problem of two irregularly-shaped inclusions with internal uniform anti-plane stresses when the two inclusions only undergo uniform anti-plane eigenstrains.

2. Optimum design of the irregular shapes of two inclusions

We consider an infinite matrix reinforced by two interacting elastic inclusions of unknown irregular shape under a uniform change in temperature. Let S_1, S_2 and S_3 denote the left inclusion, the matrix and the right inclusion, respectively, which are perfectly bonded through the left and the right interface L_1 and L_3, as shown in Fig. 1. Throughout the paper, the subscripts 1, 2 and 3 are used to identify the associated quantities in S_1, S_2 and S_3, respectively.

For plane deformations of an isotropic elastic material, the in-plane displacements u and v, the two resultant forces f_x and f_y, and the in-plane stresses σ_{xx}, σ_{yy} and σ_{xy}, can be expressed in terms of two analytic functions $\phi(z)$ and $\psi(z)$ of the complex variable $z = x + iy$ as (Muskhelishvili, 1953)

$$\begin{aligned} &2\mu(u+\mathrm{i}v) = \kappa\phi(z) - z\overline{\phi'(z)} - \overline{\psi(z)}, \\ &f_x + \mathrm{i}f_y = -\mathrm{i}[\phi(z) + z\overline{\phi'(z)} + \overline{\psi(z)}], \end{aligned} \tag{1}$$

$$\begin{aligned} &\sigma_{xx} + \sigma_{yy} = 2[\phi'(z) + \overline{\phi'(z)}], \\ &\sigma_{yy} - \sigma_{xx} + 2\mathrm{i}\sigma_{xy} = 2[\bar{z}\phi''(z) + \psi'(z)], \end{aligned} \tag{2}$$

where $\kappa = 3 - 4v$ is for plane strain and $\kappa = (3 - v)/(1 + v)$ for plane stress; μ and v, where $\mu > 0$ and $0 \leqslant v \leqslant 0.5$, are the shear modulus and Poisson's ratio, respectively.

In the physical z-plane, the original boundary value problem has the following form

$$\begin{aligned} &\phi_2(z) + z\overline{\phi_2'(z)} + \overline{\psi_2(z)} = \phi_1(z) + z\overline{\phi_1'(z)} + \overline{\psi_1(z)}, \\ &\tfrac{1}{2\mu_2}\left[\kappa_2\phi_2(z) - z\overline{\phi_2'(z)} - \overline{\psi_2(z)}\right] \\ &\quad = \tfrac{1}{2\mu_1}\left[\kappa_1\phi_1(z) - z\overline{\phi_1'(z)} - \overline{\psi_1(z)}\right] + \delta_1 z, z \in L_1; \end{aligned} \tag{3}$$

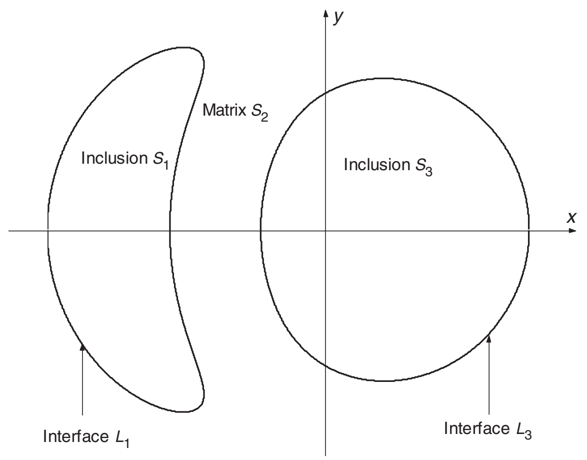

Fig. 1. Two interacting inclusions S_1 and S_3 of irregular shape with internal uniform hydrostatic thermal stresses due to a uniform change in temperature.

$$\begin{aligned} &\phi_2(z) + z\overline{\phi_2'(z)} + \overline{\psi_2(z)} = \phi_3(z) + z\overline{\phi_3'(z)} + \overline{\psi_3(z)}, \\ &\tfrac{1}{2\mu_2}\left[\kappa_2\phi_2(z) - z\overline{\phi_2'(z)} - \overline{\psi_2(z)}\right] \\ &\quad = \tfrac{1}{2\mu_3}\left[\kappa_3\phi_3(z) - z\overline{\phi_3'(z)} - \overline{\psi_3(z)}\right] + \delta_3 z, z \in L_3; \end{aligned} \tag{4}$$

$$\phi_2(z) \cong O(1), \psi_2(z) \cong O(1), \quad \text{as} \quad |z| \to \infty, \tag{5}$$

where $\delta_1 z$ and $\delta_3 z$ are the stress-free displacements caused by uniform volume eigenstrains imposed on the two irregularly shaped inclusions S_1 and S_3, respectively, which are determined by the thermal mismatches between the materials through (Ru, 1998)

$$\delta_1 = \frac{(\alpha_1 - \alpha_2)\Delta T}{2}, \quad \delta_3 = \frac{(\alpha_3 - \alpha_2)\Delta T}{2}, \tag{6}$$

with ΔT being the uniform change in temperature and α the thermal expansion coefficient.

In order to solve the boundary value problem, we introduce the following conformal mapping function:

$$z = \omega(\xi) = R\left[\frac{1}{\xi - \lambda} + \sum_{n=1}^{+\infty}(a_n\xi^n + a_{-n}\xi^{-n})\right], \quad \xi(z) = \omega^{-1}(z),$$
$$\left(1 \leqslant |\xi| \leqslant \rho^{-\frac{1}{2}}\right), \tag{7}$$

where R is a real scaling constant, $\lambda\,(1 < \lambda < \rho^{-\frac{1}{2}})$ is a positive real constant, and a_n, a_{-n} are complex constants to be determined. The above mapping function is composed of a first-order pole at $\xi = \lambda$ located within the annulus $1 \leqslant |\xi| \leqslant \rho^{-\frac{1}{2}}$ and a standard Laurent series whose coefficients are to be determined. The above mapping function will conformably map the matrix region excluding the two inclusions onto the annulus $1 \leqslant |\xi| \leqslant \rho^{-\frac{1}{2}}$. Furthermore, the left interface L_1 is mapped onto the inner unit circle $|\xi| = 1$; the right interface L_3 is mapped onto the outer circle $|\xi| = \rho^{-\frac{1}{2}} > 1$; meanwhile the point at infinity $z = \infty$ is mapped onto the point $\xi = \lambda$ within the annulus, as shown in Fig. 2. For convenience, we will write $\phi_i(\xi) = \phi_i(\omega(\xi))$ and $\psi_i(\xi) = \psi_i(\omega(\xi))$ $(i = 1, 2, 3)$.

In the ξ-plane, the boundary value problem takes the following form

$$\begin{aligned} &\phi_2(\xi) + \omega(\xi)\frac{\overline{\phi_2'(\xi)}}{\overline{\omega'(\xi)}} + \overline{\psi_2(\xi)} = \phi_1(\xi) + \omega(\xi)\frac{\overline{\phi_1'(\xi)}}{\overline{\omega'(\xi)}} + \overline{\psi_1(\xi)}, \\ &\kappa_2\phi_2(\xi) - \omega(\xi)\frac{\overline{\phi_2'(\xi)}}{\overline{\omega'(\xi)}} - \overline{\psi_2(\xi)} = \frac{\kappa_1}{\Gamma_1}\phi_1(\xi) \\ &-\frac{1}{\Gamma_1}\omega(\xi)\frac{\overline{\phi_1'(\xi)}}{\overline{\omega'(\xi)}} - \frac{1}{\Gamma_1}\overline{\psi_1(\xi)} + 2\mu_2\delta_1\omega(\xi), \quad \text{on } |\xi| = 1, \end{aligned} \tag{8}$$

$$\begin{aligned} &\phi_2(\xi) + \omega(\xi)\frac{\overline{\phi_2'(\xi)}}{\overline{\omega'(\xi)}} + \overline{\psi_2(\xi)} = \phi_3(\xi) + \omega(\xi)\frac{\overline{\phi_3'(\xi)}}{\overline{\omega'(\xi)}} + \overline{\psi_3(\xi)}, \\ &\kappa_2\phi_2(\xi) - \omega(\xi)\frac{\overline{\phi_2'(\xi)}}{\overline{\omega'(\xi)}} - \overline{\psi_2(\xi)} = \frac{\kappa_3}{\Gamma_3}\phi_3(\xi) - \frac{1}{\Gamma_3}\omega(\xi)\frac{\overline{\phi_3'(\xi)}}{\overline{\omega'(\xi)}} \\ &-\frac{1}{\Gamma_3}\overline{\psi_3(\xi)} + 2\mu_2\delta_3\omega(\xi), \quad \text{on } |\xi| = \rho^{-\frac{1}{2}}, \end{aligned} \tag{9}$$

$$\phi_2(\xi) \quad \text{and} \quad \psi_2(\xi) \text{ are analytic at } \xi = \lambda, \tag{10}$$

where $\Gamma_1 = \mu_1/\mu_2$ and $\Gamma_3 = \mu_3/\mu_2$ are two stiffness ratios.