Mechanics Research Communications 92 (2018) 87–91

Contents lists available at ScienceDirect

Mechanics Research Communications

journal homepage: www.elsevier.com/locate/mechrescom

An a priori upper bound estimate for conduction heat transfer problems with temperature-dependent thermal conductivity

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ARTICLE INFO

Article history:
Received 30 May 2018
Revised 1 August 2018
Accepted 2 August 2018
Available online 6 August 2018

Keywords:
Nonlinear heat conduction
Piecewise constant temperature-dependent thermal conductivity
Upper bound estimate
Kirchhoff transformation
Nonlinear boundary conditions

ABSTRACT

This article presents an a priori upper bound estimate for the steady-state temperature distribution in a body with any temperature-dependent thermal conductivity, generalizing a previous result (Gama et al., 2013) [1]. The discussion is carried out assuming a large class of nonlinear boundary conditions (for instance representing thermal radiant interchange). These estimates consist of a powerful tool that may avoid an expensive numerical simulation of a nonlinear heat transfer problem, whenever it suffices to know the highest temperature. In these cases the methodology proposed in this work is more effective than the usual approximations that assume thermal conductivities and heat sources as constants.

1. Introduction

The conduction heat transfer phenomenon consists of a classical issue [2–4] which, in the last years, are becoming each more time interesting due to the models arising from new engineering applications, that require more precise descriptions, in general nonlinear and, consequently, mathematically more complexes.

Problems involving temperature-dependent thermal conductivity are increasingly being studied due to the crescent use of materials which present strong dependence between the thermal conductivity and the temperature as well as due to the search of more realistic results. Such fact is observed in recent papers as, for instance, in references [5–9] where the relationship between the thermal conductivity and the temperature is the main subject, in references [10–12] where some simulations are carried out and in references [13], where the Kirchhoff transformation is the main subject.

In reference [15] it was presented an a priori upper bound estimate for conduction heat transfer problems, with linear boundary conditions, in which the thermal conductivity is assumed to be a constant. In reference [1] it is presented an a priori upper bound estimate for temperature-dependent conduction heat transfer problems, with linear boundary conditions, in which the thermal conductivity is approximated by

$$k = \begin{cases} k_1 = \text{constant for } T_0 < T \\ k_2 = \text{constant for } T_0 \geq T \end{cases} \tag{1}$$

in which T_0 is a conveniently chosen constant. The main objective of this work is to provide an upper bound estimate, suitable for a large class of nonlinear steady-state heat transfer problems (which includes all the cases considered in references [1,3]), adding the following improvements:

- Instead of [1], this work admits a more general representation for k;
- the boundary conditions need not to be linear;
- the source term need not to be bounded, provided it belongs to L^2;
- it is presented a general closed-form expression for the Kirchhoff transformation.

Eq. (1) represents an interesting, but limited, first approximation for problems with temperature-dependent thermal conductivity.

Let us begin with the mathematical description below [1,3,4]

$$\begin{aligned} div(k\,grad T) + \dot{q} = 0 \quad &\text{in} \quad \Omega \\ -k\,grad T \cdot \mathbf{n} = h(T - T_\infty) \quad &\text{on} \quad \partial\Omega \end{aligned} \tag{2}$$

which describes the classical steady-state conduction heat transfer process in a rigid and opaque body at rest, represented by the bounded open set Ω with boundary $\partial\Omega$, subjected to a linear boundary condition [2,15]. In problem (2) $\mathbf{n}$ is the unit outward

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https://doi.org/10.1016/j.mechrescom.2018.08.002
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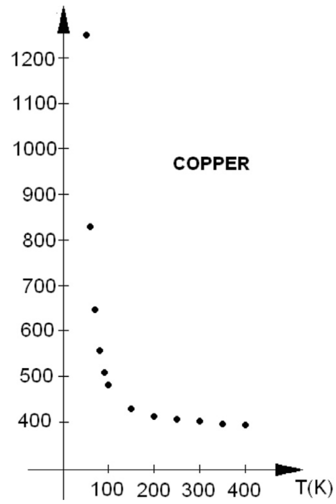

T(K)	k(W/mK)
10	24300
20	10800
30	4450
40	2170
50	1250
60	829
70	647
80	557
90	508
100	482
200	413
300	401
400	393

Fig. 1. Thermal conductivity of the copper at 1 atm.

normal, $\dot{q}$ is the internal heat generation (per unit time and unit volume), k is the thermal conductivity, T_∞ is the temperature of the environment, h is the convection heat transfer coefficient and T is the (unknown) temperature.

In this paper we consider the following generalization of Eq. (1)

$$k = \hat{k}(T) = \begin{cases} k_1 \text{ for } T_1 \geq T \\ k_i \text{ for } T_i \geq T > T_{i-1}, \quad i = 2, 3, 4, \ldots, N-1 \\ k_N \text{ for } \infty > T > T_{N-1} \end{cases} \tag{3}$$

in which k_1,k_2,..., k_N and T_1,T_2,..., T_{N-1} are positive constants (the temperatures assumed to be absolute ones). Eq. (3) is always an adequate representation and has no limit of accuracy. In addition, provides closed-form formulas for both the Kirchhoff transformation and its inverse.

An interesting comparison between Eqs. (1) and (3) takes place when it is considered the thermal conductivity of the copper as a function of the temperature [14] as presented in Fig. 1 below.

In addition, instead of problem (2), we consider the following one (more general)

$$\begin{aligned} & div(k\,grad\,T) + \dot{q} = 0 \text{ in } \Omega \\ & -k\, grad\,T \cdot \mathbf{n} = F \;\text{ on }\; \partial\Omega,\; F = \widehat{F}(T, \mathbf{x}) \end{aligned} \tag{4}$$

in which, for any $\mathbf{x} \in \partial\Omega$, there exists a positive constant δ such that

$$\frac{\partial F}{\partial T} \geq \delta > 0 \tag{5}$$

The function F could be given, for instance, by

$$\begin{aligned} & a)\; F = h\,(T - T_\infty) \quad \text{Newton law of cooling (convection)} \\ & b)\; F = \sigma T^4 - s \quad \text{Black surface with external radiant source} \\ & c)\; F = h\,(T - T_\infty) + \sigma T^4 - s \quad \text{Convection and radiation} \end{aligned} \tag{6}$$

2. The Kirchhoff transformation

The Kirchhoff transformation may be defined as follows [3]

$$\omega = \hat{\omega}(T) = \int_0^T \hat{k}(\xi)\, d\xi \tag{7}$$

Therefore

$$grad\,\omega = k\; grad\,T \tag{8}$$

and, hence,

$$div(grad\,\omega) + \dot{q} = 0 \quad \text{in} \quad \Omega \tag{9}$$

Eqs. (3) and (7) yield (after some calculations)

$$\omega = \hat{\omega}(T) = \frac{1}{2}\left((k_1 + k_N)T + \sum_{i=2}^{N} (k_i - k_{i-1})(|T - T_{i-1}| - T_{i-1}) \right) \tag{10}$$

in which $T_N > T_{N-1} > T_{N-2} > \ldots > T_3 > T_2 > T_1 \geq T_0 = 0$.

Defining the (nonnegative) constants $\omega_1, \omega_2, \omega_3, \ldots\omega_{N-2}, \omega_{N-1}$ as follows

$$\omega_i = \sum_{j=1}^{i} k_j\left(T_j - T_{j-1}\right) = \omega_{i-1} + k_i(T_i - T_{i-1}), \quad i = 1, 2, 3, \ldots, N-1, \quad with \quad \omega_0 = 0 \tag{11}$$

the inverse of the above Kirchhoff transformation can be easily obtained from the closed-form formula below

$$T = \hat{T}(\omega) = \frac{1}{2}\left(\left(\frac{1}{k_1} + \frac{1}{k_N}\right)\omega + \sum_{i=2}^{N} \left(\frac{1}{k_i} - \frac{1}{k_{i-1}}\right)(|\omega - \omega_{i-1}| - \omega_{i-1}) \right) \tag{12}$$

The positiveness of the thermal conductivity ensures that ω is an increasing function of T, while T is an increasing function of ω. Eqs. (10)–(12) require a minimum of computational effort, since they involve only linear functions and "absolute value of". There is no $\exp(\alpha T)$, $\exp(\beta\omega)$, T^α, ω^α, sines, …

In addition, the above functional relationships have no mathematical limitations. In other words, for any given $T \in (-\infty, \infty)$ Eq. (10) provides one, and only one, ω which, inserted in (12), recovers the value of T originally inserted in (10). For instance, such feature is not found when k is assumed a linear function of the temperature.

3. Estimating an upper bound for ω

The resulting problem, in terms of ω, is given by

$$\begin{aligned} & div(grad\,\omega) + \dot{q} = 0 \text{ in } \Omega \\ & -(grad\,\omega) \cdot \mathbf{n} = G \text{ on } \partial\Omega,\, G = \hat{G}(\omega, \mathbf{x}) \text{ on } \partial\Omega \\ & G = \hat{G}(\omega, \mathbf{x}) \\ & = \hat{F}\left(\left(\frac{1}{2k_1} + \frac{1}{2k_N}\right)\omega + \sum_{i=2}^{N} \left(\frac{1}{2k_i} - \frac{1}{2k_{i-1}}\right)(|\omega - \omega_{i-1}| - \omega_{i-1}), \mathbf{x} \right) \end{aligned} \tag{13}$$

At this point, let us introduce a field Ψ (there exist infinitely many) in order to satisfy the inequality

$$div(grad\,\Psi) + \dot{Q} \leq 0 \text{ in } \Omega \tag{14}$$

in which $\dot{Q}$ must be such that

$$\dot{Q} \geq \dot{q} \text{ in } \Omega \tag{15}$$

Thus,

$$\begin{aligned} & div[grad\,(\omega - \Psi)] \geq 0 \quad \text{in} \quad \Omega \\ & -grad\,(\omega - \Psi) \cdot \mathbf{n} = G + grad\,\Psi \cdot \mathbf{n} \quad \text{on} \quad \partial\Omega \end{aligned} \tag{16}$$

Since

$$div[grad\,(\omega - \Psi)] \geq 0 \quad \text{in} \quad \Omega \tag{17}$$

the divergence theorem, for any subset $\Gamma \subseteq \Omega$ with boundary $\partial\Gamma$, enables us to write

$$\int_{\partial\Gamma} grad\,(\omega - \Psi) \cdot \mathbf{n}\, dS \geq 0 \tag{18}$$