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Spectral stability of the bifurcation state of an arterial model with perivascular soft tissues

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ABSTRACT

We model an artery with perivascular soft tissue as a uniform cylindrical membrane tube surrounded by a flexible substrate with distributed stiffness. We derive the equations of motion of the arterial model, and obtain evolution equation derived in the long wavelength limit from the general equations of motion. We analyze the stability of axisymmetric perturbations at the bifurcation state taking into the consideration of surrounding soft tissue stiffness and constant axial stretch. We observe that the surrounding soft tissues progressively reduce the domain of real valued solutions with increasing constant axial stretch. The results suggest that the stationary solitary wave solution is unstable.

1. Introduction

Modeling of aneurysms, a localized bulge in the human circulatory system, and controlling its growth present significant challenges. Mathematically, an aneurysm is a bifurcation of solution of an arterial tube under uniform pressurization and stretching. Such bifurcations are a type of instability caused by dilation and weakening of biological soft tissues. In an elastomeric tube under pressurization, there can exists a local maximum in the pressure versus stretch curve, which is known as the limit-point instability of the elastomer material model [1,2]. A change in the natural global shape is referred to as geometric instability or bifurcation [3,4].

Many researchers have studied bifurcations and analyzed various instability modes in the cylindrical membrane tube [5–10]. The solution at bifurcation exhibits a radial protrusion up to a critical point beyond which the protrusion front propagates (broadens) in both directions of the tube. The competition between radial expansion and axial propagation plays an important role in evolution of aneurysm [11,12]. The growth of bifurcation of the fluid-filled membrane tube can be described through an evolution equation. The evolution equation of a model of fluid-filled membrane tube about the bifurcation state is a Boussinesq equation with an unstable stationary solitary wave solution [13,14]. Interestingly, consideration of fluid inertia of an inviscid fluid decreases the growth rate of the unstable mode [14]. Il'Ichev et al. [15] found that an arterial tube requires high pressure for onset of bifurcation when there is no imperfection in the membrane tube.

Modeling and study of the effect of perivascular tissues on an arterial tube has not been considered in most of the previous studies. Aneurysm formation and interaction with surrounding tissues have been modeled in different ways [16–18]. Varatharajan et al. [19], have observed that a neo-Hookean arterial model exhibits a delay in the onset of bifurcation and a subsequent subcritical jump in the circular distension at bifurcation with increasing perivascular substrate stiffness. Further, an anisotropic neo-Hookean arterial model can change the subcritical jump to a supercritical jump.

An effective model considering the properties of the arterial tube with surrounding soft tissues remains unexplored. We study the spectral stability of a local bulge in the perivascular arterial tube to understand the effect of surrounding soft tissues as a continuation of previous work [19].

In this article, the equations of motion of a perivascular arterial model are derived in Section 2. An evolution equation and its solution are derived in the long wavelength limit from the general equations of motion in Section 3. In Section 4, the effect of stiffness of the surrounding soft tissues on the arterial tube and its spectral stability have been analyzed.

2. Kinematics of deformation

Fig. 1 shows the schematic representation of an aorta with surrounding soft tissues. The aorta is considered to be a homogeneous, prestressed hyperelastic cylindrical membrane tube

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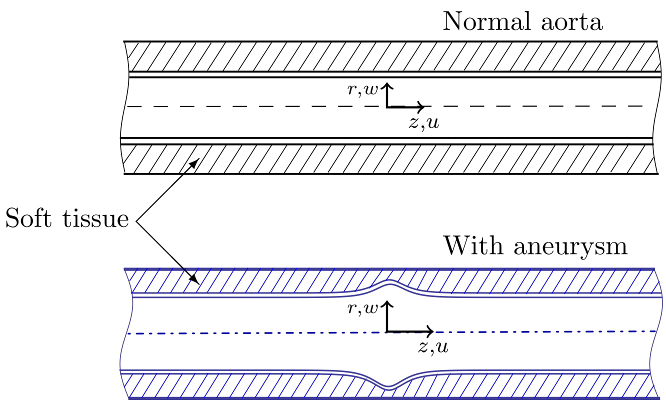

Fig. 1. Schematic representation of aorta with surrounding soft tissues.

with the reference and deformed configurations as $(\widetilde{R},\widetilde{\Theta},\widetilde{Z})$ and $(\widetilde{r},\widetilde{\theta},\widetilde{z})$, respectively. We assume that the inflation is axisymmetric, and hence, the deformation gradient for cylindrical tube is only in the principle directions which are defined by $\lambda_\theta = \frac{\widetilde{r}}{\widetilde{R}}$, $\lambda_z = \sqrt{\widetilde{r}'^2+\widetilde{z}'^2}$, $\lambda_r = \frac{\widetilde{h}}{\widetilde{H}}$. Here, $(\widetilde{H})\widetilde{h}$ is the thickness of the (undeformed) deformed cylindrical membrane tube and $(\,\cdot\,)'$ is $\frac{d}{d\widetilde{Z}}$. The strain tensor invariants of the right Cauchy-Green deformation tensor are $I_1 = \lambda_\theta^2+\lambda_z^2+\lambda_r^2$, $I_2 = \frac{1}{\lambda_\theta^2}+\frac{1}{\lambda_z^2}+\frac{1}{\lambda_r^2}$ and $I_3 \equiv \lambda_\theta\lambda_z\lambda_r = 1$.

The kinetic energy is given by

$$T = 2\pi\widetilde{R}\widetilde{H}\int_{\widetilde{L}}\frac{1}{2}\widetilde{\rho}(\dot{\widetilde{r}}^2+\dot{\widetilde{z}}^2)d\widetilde{Z}, \tag{1}$$

where $\widetilde{\rho}$ is the density of the material, "overdot" is $\frac{d}{d\widetilde{t}}$, and $\widetilde{L}$ is the length of the cylinder. The strain energy for the cylindrical membrane is given by

$$\widetilde{U}_s = 2\pi\widetilde{R}\widetilde{H}\int_{\widetilde{L}}\widehat{W}d\widetilde{Z}, \tag{2}$$

where $\widehat{W}(\lambda_\theta,\lambda_z,\lambda_r) \equiv \widetilde{W}(\lambda_\theta,\lambda_z)$ (using incompressibility property). We consider the neo-Hookean strain energy density function $\widetilde{W} = \frac{\mu}{2}[(I_1-3)]$, where $\mu > 0$ is the shear modulus [20,21].

The potential energy of the inflating fluid is given by $\widetilde{U}_g = -\widetilde{P}_g\pi\int_{\widetilde{L}}\widetilde{r}^2\widetilde{z}'d\widetilde{Z}$, $\widetilde{P}_g$ is the ideal fluid pressure which is inside the membrane tube. The pressure due to the elastic substrate is given by $\widetilde{P}_f = \widetilde{k}\widetilde{w}$, where $\widetilde{k}$ is the linear stiffness of the substrate and $\widetilde{w} = \widetilde{r}-\widetilde{R}$. The transmural pressure $\widetilde{P}_d$, can be obtained as $\widetilde{P}_g - \widetilde{P}_f = \widetilde{P}_d$. Then, $\widetilde{P}_g = \widetilde{k}\left(\widetilde{r}-\widetilde{R}\right) + \frac{\widetilde{H}}{\widetilde{R}}\frac{\widetilde{W}_{\lambda_\theta}}{\lambda_\theta\lambda_z}$. The strain energy of elastic substrate [22] is

$$\widetilde{U}_f = \pi\widetilde{k}\int_{\widetilde{L}}(\widetilde{r}-\widetilde{R})^2\widetilde{r}\widetilde{z}'d\widetilde{Z}. \tag{3}$$

Here, energy densities are defined over per unit volume in the reference configuration.

Extremizing the variational integral of the Lagrangian $\mathrm{L} = T - (\widetilde{U}_g+\widetilde{P}_g+\widetilde{U}_f)$ gives the equations of motion as

$$\widetilde{\rho}\ddot{\widetilde{r}} = \left(\frac{\widetilde{W}_{\lambda_z}\widetilde{r}'}{\lambda_z}\right)' - \frac{\widetilde{W}_{\lambda_\theta}}{\widetilde{R}} + \frac{\widetilde{P}_g}{\widetilde{R}\widetilde{H}}\widetilde{r}\widetilde{z}' - \frac{\widetilde{k}}{\widetilde{R}\widetilde{H}}\left(\widetilde{r}-\widetilde{R}\right)\widetilde{r}\widetilde{z}' - \frac{\widetilde{k}}{2\widetilde{R}\widetilde{H}}\left(\widetilde{r}-\widetilde{R}\right)^2\widetilde{z}',$$

$$\widetilde{\rho}\ddot{\widetilde{z}} = \left(\frac{\widetilde{W}_{\lambda_z}\widetilde{z}'}{\lambda_z}\right)' - \frac{\widetilde{P}_g}{\widetilde{R}\widetilde{H}}\widetilde{r}\widetilde{r}' + \frac{\widetilde{k}}{\widetilde{R}\widetilde{H}}\left(\widetilde{r}-\widetilde{R}\right)\widetilde{r}\widetilde{r}' + \frac{\widetilde{k}}{2\widetilde{R}\widetilde{H}}\left(\widetilde{r}-\widetilde{R}\right)^2\widetilde{r}',$$

where $\widetilde{W}_{\lambda_\theta} = \frac{\partial\widetilde{W}}{\partial\lambda_\theta}$, and $\widetilde{W}_{\lambda_z} = \frac{\partial\widetilde{W}}{\partial\lambda_z}$.

The expression for fluid pressure which cause the bifurcation can be derived from the conservation of mass and the linear momentum equations [14,23]. The density of the fluid is $\widetilde{\rho}_f$ (constant), velocity of the fluid is $\widetilde{v}_f$, and the viscosity is neglected [24]. The equation of mass conservation per unit area $A = \pi\widetilde{r}^2$ may be given by

$$\frac{\partial\widetilde{r}}{\partial\widetilde{t}} + \widetilde{v}_f\frac{\partial\widetilde{r}}{\partial\widetilde{z}} + \frac{\widetilde{r}}{2}\frac{\partial\widetilde{v}_f}{\partial\widetilde{z}} = 0. \tag{4}$$

From the linear momentum, we have

$$\frac{\partial\widetilde{v}_f}{\partial\widetilde{t}} + \widetilde{v}_f\frac{\partial\widetilde{v}_f}{\partial\widetilde{z}} + \frac{1}{\widetilde{\rho}_f}\frac{\partial\widetilde{P}_g}{\partial\widetilde{z}} = 0. \tag{5}$$

Suppose that $\xi(\widetilde{z},\widetilde{t})$ is a dynamical variable and let the fixed position $\widetilde{z}$ in the spatial coordinate to material coordinate $\widetilde{Z}$ be represented as $\widetilde{z} = \widetilde{z}(\widetilde{Z},\widetilde{t})$. The derivatives of spatial coordinates becomes

$$\frac{\partial\xi(\widetilde{z},\widetilde{t})}{\partial\widetilde{z}} = \frac{1}{\widetilde{z}'}\frac{\partial\xi(\widetilde{z},\widetilde{t})}{\partial\widetilde{Z}} \tag{6}$$

$$\frac{\partial\xi(\widetilde{z},\widetilde{t})}{\partial\widetilde{t}} = \frac{1}{\widetilde{z}'}\left(\widetilde{z}'\frac{\partial\xi(\widetilde{Z},\widetilde{t})}{\partial\widetilde{t}} - \frac{\partial\xi(\widetilde{Z},\widetilde{t})}{\partial\widetilde{Z}}\dot{\widetilde{z}}\right). \tag{7}$$

For dimensionless variables, we consider $\widetilde{z}\to\widetilde{R}z$, $\widetilde{r}\to\widetilde{R}r$, $\widetilde{Z}\to\widetilde{R}Z$, $\widetilde{H}\to\widetilde{R}H$, $\frac{\mu}{\widetilde{\rho}_f}\to\nu^2$, $\widetilde{t}\to\frac{\widetilde{R}}{\nu}t$, $\widetilde{v}_f\to\nu v$, $\widetilde{P}_g\to\mu P_g$, $\widetilde{k}\to\frac{\mu}{\widetilde{R}}K$, $\frac{\widetilde{\rho}}{\widetilde{\rho}_f}\to\rho$. Applying the transformations, (6) and (7) in (4) and (5) (H is taken as unity), we have the equations of motion

$$\rho\ddot{r} = \left(\frac{W_{\lambda_z}}{\lambda_z}r'\right)' - W_{\lambda_\theta} + P_g rz' - K(r-1)rz' - \frac{K}{2}(r-1)^2z' \tag{8}$$

$$\rho\ddot{z} = \left(\frac{W_{\lambda_z}}{\lambda_z}z'\right)' - P_g rr' + K(r-1)rr' + \frac{K}{2}(r-1)^2r' \tag{9}$$

$$\dot{r}z' - r'\dot{z} + vr' + \frac{r}{2}v' = 0 \tag{10}$$

$$\dot{v}z' - v'\dot{z} + vv' + P_g' = 0 \tag{11}$$

3. Evolution equation and its solution

Since the exact solution of the full nonlinear system (8)–(11) is not known, a perturbation method has been applied to analyze the solution. We consider the multiple scale expansion [13,14] to derive evolution equation using the assumption of long wavelength limit from the general equations of motion (8)–(11). We perturb uniform cylindrical tube to cylindrical tube with bulging. The strain energy density $W(\lambda_\theta,\ \lambda_z)$ is an analytic function and can be expanded using Taylor's series. The dispersion relation of the system (8)–(11) may be represented in the long wavelength limit as $c = C_0 + C_1k^2 + C_2k^4 + \ldots$, where C_0, C_1, C_2, ... are appropriate coefficients in the expansion, $c \in \mathbb{R}$ is the wave speed, and $k > 0$ is the wave number. Assuming that the fluid is stationary before the propagation of the disturbance, $C_0 = 0$.

Fu et al. [25] found that the wave speed of the slowest wave can be determined by $c^2 \sim \omega \sim (r_\infty - r_0)$ if $v_{f\infty} = 0$, where ω is the frequency, r_∞ is radius of the membrane tube at infinity, r_0 is radius at bifurcation and $v_{f\infty}$ is velocity of the fluid at infinity. In a long wavelength limit, one may express $k^2 = \epsilon\varsigma^2$. Hence, the long spatial variable and slow time variable can be assumed as $\zeta = \epsilon^{1/2}Z$ and $\tau = \epsilon t$, respectively [13]. It has been also observed that the ratio of radial and axial displacements is $O(\sqrt{\epsilon})$ [6,13,26].