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Some remarks about a simple history dependent nonlinear viscoelastic model

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ABSTRACT

A simple model for history dependent nonlinear viscoelasticity is considered. The determining equation governing shear motions is derived and investigated in the quasistatic approximation and under the traveling waves ansatz. Traveling waves are possible only if an inequality involving the constitutive parameters is satisfied. This fact is in contrast to what happens in viscoelasticity of the Kelvin–Voigt type. On the other hand, in the quasi-static approximation (classical creep and recovery experiments) the behavior of the history dependent model is similar to analogous rate dependent models.

1. Introduction

Three-dimensional theory of nonlinear viscoelasticity is an important but very complex subject.

For example, viscous dampers are ubiquitous components of many mechanical applications where vibrations need to be damped, isolated or controlled. In *empirical* finite degree models viscous dampers are considered via a *dashpot* element, i.e. by the action of a velocity dependent force in a direction opposing the velocity of the vibrating mass.

The basic Kelvin–Voigt model of linear viscoelasticity is the three-dimensional version of a purely viscous linear damper and linear elastic spring connected in parallel.

Generalizations of the Kelvin–Voigt model to a linear and nonlinear setting has been provided by many authors. For example, a possible generalization suitable for the linear setting is contained in the celebrated elasticity book by Landau and Lifchitz (more precisely: at the end of Section 34, Chapter 5) [14]. The extension of this proposal in a nonlinear setting is not straightforward. For this reason, many nonlinear viscoelastic models based on this idea are incorrect [1]. The details of the history and problems for the nonlinear version of the Kelvin–Voigt model have been discussed by Destrade, Saccomandi and Vianello in [5] where a correct (i.e. frame indifferent) generalization to nonlinear materials of the Landau and Lifchitz model has been provided.

For the nonlinear Kelvin–Voigt model in recent times several results have been obtained (see [3,11,17]). These results show that when the *constant* Kelvin–Voigt *viscosity* is replaced by a constitutive function (of the amount of strain and/or of the amount of the strain rate) several mathematical complexities can be encountered.

To ensure a meaningful mechanical behavior of a Kelvin–Voigt mathematical model with a non-constant viscosity we need: (i) a well-defined linear limit of the nonlinear viscosity function (see for example [15]); (ii) the boundedness from above and from below of the function (see for example [16]).

If in an empirical finite dimensional model we connect a purely viscous damper and a purely elastic spring in series we obtain a Maxwell model. This is a model quite different from the Kelvin–Voigt model because it takes into account *stress relaxation* and not strain relaxation. The generalization of the Maxwell model to large deformations and in a general three-dimensional setting is complicated by the use of objective derivatives with respect time. Whereas in fluid-mechanics many models in this class of materials have been proposed, in solid mechanics this idea have been investigated only in few occasions [6,20].

Using elastomer components in engineering applications stress relaxation, creep and the recovery of set after a period of constant loading are experimentally detected. In a paper, about cross-linked unfilled natural rubber, by Alan Gent [9] (see also [19]) the relationships between various manifestations of viscoelastic behavior have been experimentally examined into details. The main findings are that:

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- the stress relaxation rate is substantially independent of the amount or the type of the deformation for moderate, but finite, deformations;
- for creep and recovery a linear dependence on the logarithm of time is observed;
- the rates of the creep and the recovery behavior are related to the nonlinear stress–strain behavior of the elastomer.

This means that the model proposed in [6] can be effectively used to describe stress relaxation behavior in natural rubber, but we need to be very careful in the choice of the dissipative part of the stress tensor associated with the history of strain to reproduce the creep and recovery data.

It is well known that dissipative phenomena can be not only described by a Kelvin–Voigt model, but they can also be characterized by using forces dependent on the positions that the object occupied (in relation to its present position), a period of time Δt before the current time. These actions are denoted history dependent dissipative forces. In this framework stresses in the material depend on past as well as present states of deformation. This idea has been first proposed in a linear framework by Boltzmann in 1876. A nonlinear version of this model have been proposed in [2], investigated in detail in [7] and applied to modeling shear mountings in [8].

The aim of this note is to investigate the determining equation for shear motions in the framework of this history dependent nonlinear constitutive model to point out some basic mathematical features with the aim of developing feasible and effective models for natural rubber. To this end we investigate the possibility of traveling waves and the creep and recovery phenomena in the quasi-static approximation. For the sake of mathematical simplicity we start considering a model that does not take into account stress relaxation. The model, we are considering can be easily implemented in a constitutive law with stress-relaxation like the one considered in [6].

The qualitative theory of one-dimensional motions for integro-differential models have been investigated by the methods of functional analysis by several authors (see for example the review paper [12] and the note [10]), but here we are interested in determining analytical closed form solutions for (smooth) traveling waves and quasi-static motions to provide a direct and simple interpretation of various constitutive requirements. This is because in a nonlinear framework the dissipative properties of the material can depend on the strain and/or strain rate and therefore the constitutive choices are a delicate matter.

For an introductory review to the nonlinear theory of viscoelasticity we refer the reader to [18].

2. Basic equations

Let $\mathcal{R}$ denote a fixed reference configuration of a body $\mathcal{B}$ and let introduce the Cartesian coordinates $\boldsymbol{X}$ to identify each particle of the body. A motion assign a $\boldsymbol{x}=\boldsymbol{\chi}(\boldsymbol{X},t)$ to each particle $\boldsymbol{X}$ at each instant of time t. If the symmetric Cauchy stress tensor is denoted as $\boldsymbol{T}$ the balance of linear momentum, in the absence of body forces, is given by

$$\rho \boldsymbol{x}_{tt} = \operatorname{div} \boldsymbol{T}, \tag{2.1}$$

where the operator div is the divergence with respect to $\boldsymbol{x}$, a subscript is the partial derivative with respect to the indicated independent variable and ρ is the material density in the current configuration.

Let us introduce the right Cauchy–Green deformation tensor $\boldsymbol{C} = \boldsymbol{F}^T \boldsymbol{F}$ (equivalently of the left Cauchy–Green deformation tensor $\boldsymbol{B} = \boldsymbol{F}\boldsymbol{F}^T$), $\boldsymbol{F}$ being the deformation gradient tensor relative to the (unstressed) reference configuration; thus the principal invariants are

$$I_1 = \operatorname{tr}\boldsymbol{C}, \quad I_2 = \frac{1}{2}[(\operatorname{tr}\boldsymbol{C})^2 - \operatorname{tr}(\boldsymbol{C}^2)], \quad I_3 = \det \boldsymbol{C}. \tag{2.2}$$

We restrict our attention to isotropic and incompressible materials. Therefore, only isochoric deformations, $J := I_3^{1/2} := \det \boldsymbol{F} = 1$, are possible.

We split the Cauchy stress tensor in a elastic and a dissipative part: $\boldsymbol{T} = \boldsymbol{T}^e + \boldsymbol{T}^d$.

2.1. Elastic part

For the elastic part we introduce the strain-energy density $W = W(I_1, I_2)$ such that

$$\boldsymbol{T}^e = -p\boldsymbol{I} + 2W_1\boldsymbol{B} - 2W_2\boldsymbol{B}^{-1}, \tag{2.3}$$

where $\boldsymbol{I}$ is the identity tensor, $W_i = \partial W/\partial I_i$ $(i = 1, 2)$ and p is the Lagrange multiplier associated with the isochoricity constraint.

An example of a strain energy-density function is given by

$$W = \frac{\mu}{2}\left[(I_1 - 3) + \frac{\beta}{2}(I_1 - 3)^2\right]. \tag{2.4}$$

Here $\mu > 0$ is the infinitesimal shear modulus and β is a (positive) constitutive parameter. When $\beta = 0$ from (2.4) we recover the neo-Hookean strain-energy.

2.2. Dissipative part

We introduce the kinematical quantity

$$\boldsymbol{J}_t(\boldsymbol{X}, t-s) = \boldsymbol{C}_t(\boldsymbol{X}, t-s) - \boldsymbol{I}, \tag{2.5}$$

where $\boldsymbol{C}_t(\boldsymbol{X}, t-s) = \boldsymbol{F}^{-T}(\boldsymbol{X}, t)\,\boldsymbol{F}^T(\boldsymbol{X}, t-s)\,\boldsymbol{F}(\boldsymbol{X}, t-s)\,\boldsymbol{F}^{-1}(\boldsymbol{X}, t)$.

The (2.5) is used to define the dissipative part of the Cauchy stress tensor as

$$\boldsymbol{T}^d = \int_0^\infty \varphi(\boldsymbol{F}(\boldsymbol{X}, t), \boldsymbol{D}(\boldsymbol{X}, t), s)\boldsymbol{J}_t\, ds, \tag{2.6}$$

where the stretching tensor is $\boldsymbol{D} \equiv {}^1/_2(\dot{\boldsymbol{F}}\boldsymbol{F}^{-1} + \boldsymbol{F}^{-T}\dot{\boldsymbol{F}}^T)$, and we assume that the memory fades in time such that $\varphi(s) = d\Phi/ds$. The relaxation function Φ is such that $\lim_{s\to\infty} \Phi = 0$ and we consider the same choice of [7]

$$\Phi(\boldsymbol{F}(\boldsymbol{X}, t), \boldsymbol{D}(\boldsymbol{X}, t), s) = \nu(\boldsymbol{F}(\boldsymbol{X}, t), \boldsymbol{D}(\boldsymbol{X}, t)) \exp\left(\frac{-s}{\gamma}\right). \tag{2.7}$$

In the single exponential modulus the γ is a constant relaxation time.

Because we are considering isotropic materials the function ν must depend with respect to the invariants I_1, I_2, the invariants of the stretching tensor tr $\boldsymbol{D}^2$, tr $\boldsymbol{D}^3$, and the mixed invariants tr($\boldsymbol{B}\boldsymbol{D}$), tr($\boldsymbol{B}^2\boldsymbol{D}$), tr($\boldsymbol{B}\boldsymbol{D}^2$), tr($\boldsymbol{B}^2\boldsymbol{D}^2$).

2.3. Shear motions

We are interested in the shear motion

$$x = X + u(Y, t), \quad y = Y, \quad z = Z, \tag{2.8}$$

where u is the unknown displacement. If we introduce the shear strain $K = u_Y$ we compute the gradient of deformation

$$[\boldsymbol{F}]_{ij} = \begin{bmatrix} 1 & K & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$