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A mathematical model for the propulsive thrust of the thin elastic wing harmonically oscillating in a flow of non-viscous incompressible fluid

Mezhlum A. Sumbatyan*, Alexander E. Tarasov

Southern Federal University, Institute for Mathematics, Mechanics and Computer Science, Milchakova Street 8a, Rostov-on-Don 344090, Russia

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ABSTRACT

In the present paper there is proposed an analytical approach to study vibration of a rectangular elastic wing in the stationary stream of non-viscous fluid. We first develop a basic two-dimensional integral equation. Then a series expansion along the short coordinate is applied. This reduces the problem to an infinite set of one-dimensional integral equations which is studied asymptotically with respect to the large aspect ratio parameter. An example of optimization of thickness of the wing is demonstrated, to test the efficiency of the proposed method in applications.

1. Introduction

Apparently, Leonardo da Vinci was the first who investigated the question in which way flapping wings of birds provide thrust and lift forces sufficient to fly. Nevertheless, until the beginning of the 20th century the problem of wing oscillations in fluid could not be studied theoretically even in simple formulations. The thrust effect of flapping wing was explained by Knoller [1] and Betz [2], independently. Then the successor of Prandtl, Birnbaum [3] initiated a thorough study of aerodynamic forces acting to the oscillating wing. He introduced important concepts of free and bound vortices and the concept of vortex drag. Theodorsen [4] developed a transient aerodynamic theory of thin flapping airfoil. Garrick [5] further generalized this theory, to calculate the propulsive force of the flapping wing. Some technical errors in the work of Garrick were identified by Peters and Johnson [6], besides these authors give in their work a complete unsteady aerodynamic theory. An application of this theory to analyze sinusoidal locomotion was demonstrated in [7].

In 1935 Keldysh and Lavrentiev [8] for the first time applied to the problem under consideration some methods of the theory of complex-valued analytical functions. Further development of this theory was carried out by Sedov, Keldysh, Nekrasov, and Lavrentiev (see [9]). However, all investigations described above were performed only in the two-dimensional case.

The passage to the three-dimensional transient (non-stationary) theory complicates significantly the problem which, as vividly expressed by Prandtl, is "a problem of transcendental difficulty". In 1938 Cicala [10] extends Prandtl's theory of lifting vortices to the case of transient flow. Among other results of good precision and satisfactory confirmation by experimental data, one may refer to the work of Reissner [11] who applied Prandtl's method of acceleration potential to these problems.

Some time later, various experimental techniques were applied, to study flapping airfoils (see, for example, [12]). Moreover, recently a number of direct numerical approaches have been developed and applied to the discussed problems. A good survey of numerical as well as collaborative experimental-numerical methods, with further helpful references, can be found in [13,14]. It should be noted that many recent works pay more attention to practical aspects of flight, by establishing new mechanisms in flying of birds and insects [15–18], by both theoretical and experimental techniques.

The present work is aimed at the study of harmonic oscillations of the flapping rectangular elastic wing in a stationary flow. We first develop a basic two-dimensional integral equation. Then for large aspect ratio we apply a justified asymptotic method, to reduce the equation to a set of one-dimensional integral equations. Finally,

* Corresponding author. Tel.: +7 9281397067.
E-mail addresses: sumbat@math.rsu.ru (M.A. Sumbatyan), tarasovmech@gmail.com (A.E. Tarasov).

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there is demonstrated an example of an optimization relatively to the thickness of the wing.

2. Mathematical formulation of the problem

Let a thin flat wing of the size $(-\ell,\ \ell)\times(-c,c)$, rectangular in plan, be placed into a homogeneous stream of non-viscous incompressible fluid. The wing is modeled as an elastic beam of length $2l$ with the constant bending stiffness EJ and linear mass density m. A set of parallel linear rigid chords of equal length $2c$ is joined to the beam, and the beam can only bend, so that all chords keep parallel to each other (see Fig. 1), the beam thus is absolutely rigid with respect to torsion.

Let the outer forcing be caused by harmonic oscillations of the axis of symmetry x, so that the amplitude of the central displacement W_0 and the amplitude of the central inclination angle W_1 are known. Due to the outer forcing, the oscillations propagate along the span of the wing. It is assumed that the oscillation process is symmetric with respect to y, and the attack angle is zero. We are thus interested in the mechanism of the thrust force only, ignoring lift. The perturbations introduced in the stream by the wing are assumed to be small, so that the problem is studied in linear approximation. Then all physical quantities are harmonic in time: $\tilde{A}(x,y,z,t)=\mathrm{Re}\{A(x,y,z)e^{-i\omega t}\}$.

Let $z=\tilde{W}(y,t)=\mathrm{Re}\{W(y)e^{-i\omega t}\}$ denote the function which determines the shape of the wing. In the linear approximation it is assumed that $|dW/dy|\ll 1$. If the side edges of the wing are free of load, then the boundary conditions for the oscillating elastic beam are:

$$W=W_0,\quad \frac{dW}{dy}=\pm W_1 e^{i\chi},\quad (y=\pm 0);\quad \frac{d^2W}{dy^2}=\frac{d^3W}{dy^3}=0,\quad (y=\pm\ell). \tag{2.1}$$

As indicated above, quantities W_0 and W_1 designate, respectively, the amplitude of the vertical oscillations and the angular amplitude on the axis of symmetry, and χ designates the phase shift between the angular and the vertical oscillations.

The dynamic elastic behavior of the beam itself is defined from the following differential equation:

$$EJ\frac{\partial^4\tilde{W}}{\partial y^4}+m\frac{\partial^2\tilde{W}}{\partial t^2}=\tilde{Z},\Rightarrow EJ\frac{d^4W\quad(y)}{dy^4}-m\omega^2W(y)=Z(y),\ Z(y)=\int_{-c}^{c}(p_- - p_+)dx, \tag{2.2}$$

where p_- and p_+ denote the hydrodynamic pressure distributed under and above the wing, respectively.

The form of the wing oscillations as well as all other mechanical characteristics are determined, on the one hand, by elastic properties of the wing, and on the other hand, by hydrodynamic interaction forces arising between the wing and the flow.

The linearized theory implies that perturbations of the velocity and of the pressure are small compared to their values u_0, p_0 at infinity (as $x\to-\infty$): $\bar{v}=\bar{u}_0+\bar{v}'$, $p=p_0+p'$, $|v'/u_0|\ll 1$, $|p'/p_0|\ll 1$. The hydrodynamic field is assumed to be potential in the whole 3d space, outside the wing and the vortex wake. The latter stretches along the stream to $x=+\infty$ behind the wing, from its trailing edge. This implies for the perturbed velocity vector: $\bar{v}'(x,y,z)=\mathrm{grad}\varphi$. Then the equation of continuity and the linearized Lagrange-Cauchy integral imply, respectively

$$\Delta\varphi=0;\quad \frac{\partial\tilde{\varphi}}{\partial t}+u_0\frac{\partial\tilde{\varphi}}{\partial x}=-\frac{\tilde{p}'}{\rho},\quad\Rightarrow\quad -i\omega\varphi+u_0\frac{\partial\varphi}{\partial x}=-\frac{p'}{\rho}, \tag{2.3}$$

where ρ is the mass density of the fluid.

The hydrodynamic boundary conditions take the following form. The perturbations vanish at infinity, hence the potential can be accepted vanishing at infinity: $\varphi\to 0$, $(x\to-\infty,\ z\to\pm\infty)$. It is interesting to notice that the potential does not vanish as $x\to+\infty$, because of the vortex wake discussed above, which leads to discontinuity of the tangential component of the velocity vector when crossing the vortex wake. However, the pressure and the normal component of the velocity are continuous, hence

$$p'_-=p'_+,\quad \frac{\partial\varphi_-}{\partial z}=\frac{\partial\varphi_+}{\partial z},\quad z=0,\quad (x,y)\notin \text{wing}. \tag{2.4}$$

The solid airfoil's surface implies $(W=W(y))$:

$$\frac{\partial\tilde{W}}{\partial t}+u_0\frac{\partial\tilde{W}}{\partial x}=\frac{\partial\tilde{\varphi}}{\partial z},\quad\Rightarrow -i\omega W=\frac{\partial\varphi_-}{\partial z}=\frac{\partial\varphi_+}{\partial z},\quad z=0,\ (x,y)\in\text{wing}. \tag{2.5}$$

3. The basic integral equation

Here we resolve the hydrodynamic problem. By combining a classical treatment of the method of continuous [19] and discrete [20] vortices with the two-dimensional Fourier transform along variables x and y, boundary value problem (2.3)–(2.5) will be reduced to a two-dimensional integral equation.

It follows from Eqs. (2.4) and (2.5) that $\partial\varphi/\partial z$ is even with respect to z. Therefore, potential φ is odd with respect to z for all (x,y). Then, obviously, the perturbation of the aerodynamic pressure p' in (2.3) is also odd with respect to z for all (x,y). It is thus sufficient to find these basic functions only for positive z, for example.

By applying the two-dimensional Fourier transform with respect to variables x and y, one obtains the differential equation:

$$\Phi(\alpha,\beta,z)=\int_{-\infty}^{\infty}\int\varphi(\xi,\eta,z)e^{i(\alpha\xi+\beta\eta)}d\xi d\eta,\quad \frac{\partial^2\Phi}{\partial z^2}-(\alpha^2+\beta^2)\Phi=0, \tag{3.1}$$

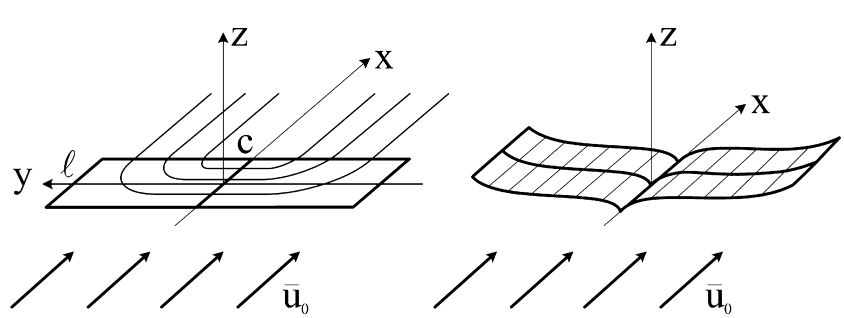

Fig. 1. Flapping elastic wing in the homogeneous flow of a non-viscous incompressible fluid.