Mechanism and Machine Theory 000 (2017) 1–13

Contents lists available at ScienceDirect

Mechanism and Machine Theory

journal homepage: www.elsevier.com/locate/mechmachtheory

Research paper

On the use of the dual Euler–Rodrigues parameters in the numerical solution of the inverse-displacement problem

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ARTICLE INFO

Article history:
Received 16 October 2017
Revised 26 November 2017
Accepted 6 December 2017
Available online xxx

Keywords:
Numerical inverse displacement problem
Dual Euler–Rodrigues parameters
Dual quaternions
Overdetermined system of nonlinear equation
Nonlinear least squares

ABSTRACT

The Euler–Rodrigues parameters (ERP) are revisited in this paper, the motivation being the need for a systematic methodology to formulate the inverse-displacement problem associated with a six-revolute (6R) serial robot. Although significant progress was made in the solution of the problem in the eighties and nineties, there is not one algorithm adopted by the research and applications communities, but rather various algorithms that lead to the resolvent polynomial of 16th degree or lower. In this paper the problem is formulated using the dual ERP. Motivated by the lack of a methodology that would allow R&D professionals to readily implement a generic solution algorithm in industrial environments requiring *real-time* inverse-displacement problem (IDP) solutions. An inverse-displacement *numerical solution* of 6R robots of arbitrary architecture is proposed. This is the main thrust of the paper. The contribution of the paper is a systematic procedure to formulate the problem and implement its solution by numerical means. The procedure is illustrated with the example of an industrial robot whose architecture does not allow for a closed-form solution.

1. Introduction

The *Euler-Rodrigues parameters* (ERP) [1,2] have received much less attention in the literature than their isomorphs, *quaternions*, although the former admit a simpler interpretation than the latter. Indeed, quaternions are usually introduced as algebraic entities on their own merit, which happen to be extremely useful in the kinematics of rotations. Indeed, quaternions allow for a compact form, with four real scalars, as opposed to the nine of the proper orthogonal matrix from which they are extracted. The ERP make up a set of *invariants* of the rotation matrix. An invariant quantity in physics is one that follows specific rules upon a change of frame [3]: if a scalar, the quantity is immutable under the change; if a vector or a tensor, then the quantity changes according to what is known as a *similarity transformation* in linear algebra [4]. A rotation is known to have what could be thought of as two *natural* invariants, one scalar, its angle of rotation, one vector, the unit vector parallel to its axis of rotation, whose direction determines the sign of the angle of rotation. Extracting these invariants from the entries of the rotation matrix is done indirectly, via an alternative set of invariants. There are basically two of

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https://doi.org/10.1016/j.mechmachtheory.2017.12.006
0094-114X/© 2017 Published by Elsevier Ltd.

these, the linear invariants and the ERP.[1] The former are the *trace* and the *axial vector* of the rotation matrix [6]. The trace is in a linear, non-homogeneous relation with the cosine of the angle of rotation, the axial vector being a bilinear function of the unit vector in the direction of the axis of rotation and the sine of the angle of rotation. Moreover, one and the same rotation matrix represents both one rotation about an axis given by the unit vector $\mathbf{e}$ through an angle ϕ, and a second, similar rotation, with the same natural invariants, but with their signs reversed. The analyst is thus free to choose either the direction of the unit vector or the sign of the angle of rotation, given that a reversal of the signs of these invariants yields the same rotation matrix. An attractive feature of the *linear invariants* is the ease with which they are extracted from the rotation matrix, as this operation is, as the name indicates, *linear*, i.e., their extraction from the rotation matrix involves only additions and subtractions of the matrix entries.

The linear invariants have been used to formulate and solve the inverse displacement problem (IDP) associated with a 6R robot numerically [7]. In the cited paper, the problem was solved using the four linear-invariant equations and the three translation equations, thereby ending up with a system of seven equations in six unknowns. Although the system is overdetermined, it is consistent, but vulnerable to a representation singularity, because the axial vector vanishes when the angle of rotation is π.

By virtue of the above singularity, linear invariants lack robustness. Here come to the rescue the ERP. The relation between these invariants and the rotation matrix is not linear, but *quadratic*. Indeed, the ERP can be interpreted as the linear invariants of the square root of the given rotation. Now, matrix square-root extraction is, in principle, a rather complex operation—it calls for an application of the Cayley–Hamilton Theorem[4], followed by linear-equation solving–but, in the case of the rotation matrix, square-rooting reduces to a simple operation: replace the angle of rotation ϕ with $\phi/2$. This is straightforward when the natural invariants are known. If they aren't, then they can be found via the linear invariants. The determination of the ERP from the linear invariants is a simple operation [6], involving only the formulas for the harmonic functions of a half angle in terms of those of the full angle. The result, the ERP, is a set of robust invariants, free of the above-mentioned singularity.

While the ERP and the quaternions are isomorphic, we prefer the former because of their direct relation with the rotation matrix. As the record shows, ERP appeared in the scholarly literature in 1840 [8], while quaternions four years later [9]. In a comprehensive article on the authorship of the concept, ERP/quaternions, Altmann [1] gives credit to O. Rodrigues, but acknowledges that very little is known of Rodrigues' life and work, while a copious literature exists on Hamilton, the creator of quaternions.

The derivation of the pertinent algebraic relations in this paper is based on the Denavit–Hartenberg (DH) notation [10]. In the balance of the paper the ERP are revisited, in their dual version, in light of the numerical solution of the IDP. Moreover, the ERP of the *i*th DH rotation matrix are derived, along with expressions for the ERP of a product of successive rotations, in Section 2. The foregoing rotation equations are complemented with their translation counterparts, which is done in Section 3. The translation equations are tersely derived from the rotation equations upon *dualizing* both the DH rotation matrices and their ERP, the desired equations being the dual parts of the foregoing dual equations. The gradient of the translation equations w.r.t. the array of joint angles is also derived upon dualizing the gradient of the rotation equations. The section ends with an overdetermined nonlinear system of eight equations in six unknowns. The solution of this system is then discussed in Section 4 by means of nonlinear least squares, as implemented with the Newton–Gauss method, which relies on the 8×6 gradient, i.e., the Jacobian matrix, of the system of equations w.r.t. the six unknown angles. The relation between this gradient and what is known in the robotics literature as "the Jacobian"[2] is discussed in this section. The implementation of the numerical solution is demonstrated with the aid of a case study in Section 5.

The motivation of the paper is thus the numerical solution of the IDP in its full generality, in which the orientation problem cannot be decoupled from the translation problem. For this reason, robots with an architecture that allows the decoupling are sometimes referred to as *decoupled robots*, which are the norm in industry at the moment. In fact, the authors can cite only two six-revolute decoupled robots in industry, the TelBot and the Fanuc Arc Mate S [6] with a non-decoupled architecture. It can be argued that one very good reason why non-decoupled robots are not popular in industry is the lack of commercial software that caters to them. One objective of this paper is to pave the way to a large variety of robots in industry, with their many advantages in terms of mobility and dexterity, among other performance indicators. One good motivation to investigate the IDP in its full generality is the acknowledgment that all robots in real life are coupled. Robots that are designed with a decoupled architecture turn out to be, after calibration, coupled. The availability of a numerical scheme that caters to coupled robots should lead to more precise *real-time* control algorithms than the state of the art. The same problem was extensively investigated in the eighties and early nineties, at about the same time that the long-standing effort for the derivation of the minimal polynomial, using elimination methods, was intense. It had been anticipated that the system of rotation and translation equations should yield a *minimal polynomial* of degree 16. The first researchers who succeeded in deriving the minimal polynomial were Raghavan and Roth [11,12] and Lee et al. [13]. These seminal papers appeared to close a chapter in the history of kinematics that led to intensive research work for over one decade all over the world. However, as recently as the past decade, Husty et al. [14] proposed a novel formulation of the IDP by means of a

[1] At least one more set of invariants has been proposed in the literature, namely, those dubbed the *modified Rodrigues parameters* [5], similar to the ERP, but with $\phi/2$ replaced with $\phi/4$. These, however, have not yet found adepts since their inception, in the mid-nineties.

[2] This putative Jacobian is the 6×6 matrix that maps the six-dimensional array of joint rates into the (six-dimensional) end-effector twist. This array is not a Jacobian, properly speaking, because the angular-velocity vector is not a time-derivative.