Mechanism and Machine Theory 000 (2017) 1–10

Contents lists available at ScienceDirect

Mechanism and Machine Theory

journal homepage: www.elsevier.com/locate/mechmachtheory

Principle of transference – An extension to hyper-dual numbers

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ARTICLE INFO

Article history:
Received 25 September 2017
Revised 19 November 2017
Accepted 5 December 2017
Available online xxx

Keywords:
Dual numbers
Hyper-dual numbers
Principle of transference

ABSTRACT

The algebra of hyper-dual numbers and hyper-dual vectors of order n, developed in this paper, follows the same rules as those of dual numbers and dual vectors. By showing that the basic formulae of vectors scalar and vector multiplication, hold for dual vectors of order n and that the basic trigonometric formulas hold for dual angles of order n, we concluded, that all formulae of vector algebra and trigonometric functions that are based on the above identities also hold for dual numbers of order n. This, as a result, extends Kotelnikov's "principle of transference" developed for dual numbers, to hyper-dual numbers of order n.

1. Introduction

The "principle of transference", which was coined by Kotelnikov more than a century ago [1], states that "All valid laws and formulae relating to a system of intersecting unit line vectors (and hence involving real variables) are equally valid to an equivalent system of skew unit line vectors, if each real variable a in the formulae is replaced by the corresponding dual variable, $\hat{a} = a + \varepsilon a_0$" [2–4].

This was shown to be true, since basic formulae of real number vectors, such as scalar and vector multiplications and trigonometric functions [5], are identical to those of dual number vectors. Hence, all formulas of vector algebra that can be reduced to these basic formulae, are also identical in dual number vectors.

The origin of Kotelnikov's work has attracted interest, due to Martinaz and Duffy's claim in their paper: "Principle of Transference: History, Statement and proof" [6], that the original manuscript was lost in the 1917 Soviet revolution. F.M. Dimnetberg [5] was one of the first to apply dual numbers and dual screws to mechanism analysis, by extensively using Kotelnikov's principle of transference. Interestingly, F.M. Dimentberg's son provided us with the original manuscript of Kotelnikov's work, which was found in his father's archive.

Dual numbers were introduced by Clifford [7] in order to expand quaternions to bi-quaternions that represent both rotations and translations.

Dual numbers

$$\hat{a} = a + \varepsilon a_0 \text{ with } \varepsilon^2 = 0;\ \varepsilon \neq 0,$$

constitute an ordered pair of real numbers, the algebra of which over $\mathbb{R}^2$ is obtained by the rule:

$$(a,\ b) \cdot (c, d) = (a\,c,\ a\,d + b\,c)$$

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https://doi.org/10.1016/j.mechmachtheory.2017.12.007

List of symbols

a	Scalar
$\hat{a}$	Dual number
$\widehat{a}$	Hyper-Dual number
$^{n}\widehat{a}$	Dual number of order n
$f(^{n}\widehat{a})$	Function of a dual number of order n
γ_i	Index shift operator
$\boldsymbol{r}$	Vector
$\hat{\boldsymbol{r}}$	Dual vector
$^{n}\widehat{\boldsymbol{r}}$	Dual vector of order n
ε	Dual unit
$\widehat{\theta}$	Dual angle
$^{n}\widehat{\theta}$	Dual angle of order n

Dual numbers were then applied to kinematics of rigid bodies (see, for example, [5,8–9]), and were further applied to the dynamics of rigid bodies (see, for example, [5,10–14]). Brodsky and Shoham [15–17] introduced the Dual Inertia Operator, which enabled derivation of the Newton-Euler and Lagrange's equations of motion in a complete three-dimensional dual form. A list of references of dual number applications in linear algebra, kinematics and numerical algorithm, can be found in [18].

Recently, hyper-dual numbers (HDN) were introduced by Fike and Alonso [19–22] and used in [23], with the aim of demonstrating the advantage of HDN in second-order numerical differentiation, as manifested by smaller numerical – subtractive and cancellation - errors as well as reduced computational time.

The general form of dual numbers of order n, DN^n, are introduced in this paper. Clifford's dual numbers [7] are referred to here as dual number of order 1, DN^1, whereas Fike and Alonso's hyper-dual numbers are referred to here as dual numbers of order 2, DN^2. It should be noted, that following this notation, real numbers are considered dual numbers of order 0, DN^0.

2. Mathematical formulation of dual numbers of order 2, DN^2

Next, the mathematical formulation of DN^2, which serves as a basis for a higher order of DN^n, is developed.

Let $\widehat{x}$ be DN^2, which consists of four real numbers (a_0, a_1, a_2, a_3) and two dual units, ε_1,ε_2, with the following multiplication rules:

$$\varepsilon_1{}^2 = \varepsilon_2{}^2 = (\varepsilon_1\varepsilon_2)^2 = 0 \tag{1}$$

$$\varepsilon_1\,,\ \ \varepsilon_2,\ \ \varepsilon_1\varepsilon_2 \neq 0 \tag{2}$$

where $\widehat{x}$ is:

$$\widehat{x} = a_0 + \varepsilon_1 a_1 + \varepsilon_2 a_2 + \varepsilon_1\varepsilon_2 a_3 \tag{3}$$

The DN^2 and the hyper-dual angle were recently applied in the field of kinematics and dynamics by Cohen and Shoham [24,25], who aimed to simplify the derivation of the equations of motion of multi-body systems. In order to gain further insight into the physical meaning of DN^2 when applied to physical entities, Cohen and Shoham [24] augmented the hyper-dual units ε_1 and ε_2 to read as follows:

$$\begin{aligned} &\varepsilon = \varepsilon_1;\ \ \varepsilon^* = \varepsilon_2 \\ &\varepsilon^2 = (\varepsilon^*)^2 = 0 \\ &\varepsilon\,,\ \varepsilon^*, \varepsilon\varepsilon^* \neq 0 \end{aligned} \tag{4}$$

Eq. (3) now takes the form of:

$$\widehat{x} = (a_0 + \varepsilon_1 a_1) + \varepsilon^*(a_2 + \varepsilon_1 a_3) \tag{5}$$

This means that a DN^2 is constituted of two lower-order dual numbers of order 1, DN^1, distinguished by the "hyper-dual unit", ε^*, just as a DN^1 is constituted of two real numbers, DN^0, distinguished by the dual unit ε.

2.1. Functions of DN^2

Taylor series expansion of a dual function of order 2, results in [19]:

$$f(\widehat{x}) = f(a_0) + \varepsilon_1 a_1 f'(a_0) + \varepsilon_2 a_2 f'(a_0) + \varepsilon_1\varepsilon_2\big(a_3 f'(a_0) + a_1 a_2 f''(a_0)\big) \tag{6}$$

Please cite this article as: A. Cohen, M. Shoham, Principle of transference – An extension to hyper-dual numbers, Mechanism and Machine Theory (2017), https://doi.org/10.1016/j.mechmachtheory.2017.12.007