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Research paper

# A novel closed-form solution for the inverse kinematics of redundant manipulators through workspace analysis

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ABSTRACT

This work addresses the inverse kinematic problem for redundant serial manipulators. Its importance relies on its effect in the programming and control of redundant robots. Besides, no general closed-form techniques have been developed so far. In this paper, redundant manipulators are reduced to non-redundant ones by selecting a set of joints, denoted *redundant joints*, and parametrizing its joint variables. This selection is made through a workspace analysis which also provides an upper bound for the number of different closed-form solutions for a given pose. Once these joints have been identified several closed-form methods developed for non-redundant manipulators can be applied for obtaining the analytical solutions. Finally, particular instances for the parametrized joints variables are determined depending on the task to be executed. Different criteria and optimization functions can be defined for that purpose.



## 1. Introduction

A serial robot manipulator is an open kinematic chain made up of a sequence of rigid bodies, called links, connected by means of kinematic pairs, called joints, that provide relative motion between consecutive links. At the end of the last link, there is a tool or device, called end-effector.

From a kinematic point of view, the end-effector position and orientation (pose) of a manipulator can be expressed as a differentiable function $f:\mathcal{C}\to X$ that relates the space of joint variables, denoted configuration space $\mathcal{C}$, with the space of all positions and orientations of the end-effector with respect to a reference frame, known as the operational space $X$. For serial manipulators, a frame used to describe the relative position and orientation is attached to each joint of the manipulator. The relations between consecutive joint frames can be expressed by homogeneous matrices based in the D-H parameters [1–4]. Therefore, each joint $i$ has associated, together with the corresponding orthonormal frame $\{\boldsymbol{o}_i, \boldsymbol{x}_i, \boldsymbol{y}_i, \boldsymbol{z}_i\}$, a homogeneous matrix that relates this frame to the precedent one (the first joint frame is related to the world frame). The function $f$, known as the kinematic function, can be represented with these homogeneous matrices. Deriving the kinematic function $f$ with respect to time, a relation in the rate domain is obtained:

$$\dot{\boldsymbol{x}} = J(\boldsymbol{q})\dot{\boldsymbol{q}} \tag{1}$$

where $\dot{\boldsymbol{x}}$ denotes the velocity vector of the end-effector; $\dot{\boldsymbol{q}}$, the vector of the joint velocities; and $J$, the Jacobian matrix associated to the manipulator. A manipulator is said to have $n$ *degrees of freedom (DOF)* if its configuration can be minimally

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specified by $n$ variables. For a serial manipulator, the number and nature of the joints determine the number of DOF. For the task of positioning and orientating its end-effector in the space, the manipulators with more than 6 DOF are called redundant while the rest are non-redundant. Redundant manipulators have $m = n - 6$ *degrees of redundancy*.

There are two types of Jacobian matrix: the geometric Jacobian $J_G(\boldsymbol{q})$ and the analytical Jacobian $J_A(\boldsymbol{q})$, depending if the last three components of $\dot{\boldsymbol{x}}$ in (1) represent the angular or the rotational velocity of the end effector, respectively. If $J_i$ denotes the $i$th column of $J_G(\boldsymbol{q})$,

$$J_i = \begin{cases} \begin{bmatrix} \boldsymbol{z}_i \times (\boldsymbol{o}_n - \boldsymbol{o}_i) \\ \boldsymbol{z}_i \end{bmatrix} & \text{if } i \text{ is revolute} \\ \begin{bmatrix} \boldsymbol{z}_i \\ 0 \end{bmatrix} & \text{if } i \text{ is prismatic} \end{cases}$$

where $\times$ denotes the cross product of two vectors in $\mathbb{R}^3$.

One of the most important kinematic problems for serial manipulators is the inverse kinematic problem. This problem consists of obtaining the joint variables, i.e. the configuration, associated to a particular pose. This configuration may not be unique, since non-redundant manipulators have up to sixteen different configurations for the same particular pose [5], while for redundant manipulators this number is unbounded [3,4]. The methods to solve the inverse kinematics problem for serial manipulators are categorized into two groups:

a) Analytical or closed-form methods: All the solutions are expressed as functions in terms of the pose.
b) Numerical methods: Starting with an initial configuration $\boldsymbol{q}_0$, an iterative process returns a good approximation $\tilde{\boldsymbol{q}}$ of one of the solutions.

The *closed-form methods* strongly depend on the geometry of the manipulator and, therefore, are not general enough. However, they are computationally efficient and give all the solutions for a given pose. In his PhD thesis, Pieper [5] develops a procedure for obtaining the closed-form solutions for a class of serial manipulators, i.e., the manipulators with three consecutive joints whose axes are either parallel or intersect at a single point (if these three consecutive joints are the last three, the robot is said to have *spherical wrist*). Later, Paul [6] establishes a more rigorous and generic method based on the handling of the homogeneous matrices that can be applied to manipulators of other kind. The main recent contributions include the use of Lagrange multipliers [7], the definition of imaginary links for redundant manipulators [8,9], the definition of the arm angle parameter [10–12] and different geometric methods [13–16].

On the other hand, *numerical methods* usually work with any manipulator, but they suffer from several drawbacks like high computational cost and execution time, existence of local minima and numerical errors. Moreover, only one of the sixteen (infinite) possible solutions is obtained for non-redundant (redundant) manipulators. The most extended numerical approaches are the Jacobian-based methods, in which the relation (1) is inverted and solved iteratively. Inverting the Jacobian matrix is not always possible. For redundant manipulators, $J(\boldsymbol{q})$ is not a square matrix while for non-redundant manipulators $\det(J(\boldsymbol{q}))$ vanishes at singularities [3,4]. To handle with these situations, alternative methods are used like pseudoinverse, transpose, damped least-squares and local optimization [3,17–28]. Other numerical methods include the use of augmented Jacobian [29], conformal geometry algebra [30,31], Crank–Nicholson methods [32] and reachability maps [33].

The importance of the inverse kinematic problem relies on its role in the programming and control of serial robots. Besides, this problem becomes of great significance for redundant manipulators because, existing an infinite number of solutions for a particular pose, manipulability measures can be defined for selecting a particular solution. Among all the methods presented in this section, closed-form ones are the most suitable for redundant manipulators as they allow to obtain the set of all solutions with a small computational cost. This paper proposes a novel method for deriving closed-form solutions for the inverse kinematics of redundant serial manipulators. These solutions are given as $m$-parameter families of functions depending on the end-effector' pose. Redundant manipulators are reduced into non-redundant ones by parametrization of a set of joint variables. These joints will be denoted as *redundant joints*. The selection of such joints is crucial and it is done using global rank deficiency conditions of the Jacobian matrix and workspace's volume analysis. An upper bound for the number of different $m$-parameter families of closed-form solutions is given. This number depends on the degrees of redundancy. Once the redundant joints are selected, the inverse kinematics of the non-redundant manipulator is solved analytically using either Pieper, Paul or other geometric methods. Finally, the particular values of the parametrized joint variables can be determined using manipulability measures or optimization criteria. The rest of the paper is organized as follows: Section 2 reviews the related work. In Section 3, the methodology is presented. This section is divided into three parts: in Section 3.1 an upper bound for the number of different closed-form solutions associated to a particular pose is given; Section 3.2 displays the different criteria for the selection of redundant joints, while in Section 3.3 several closed-form methods are used for the non-redundant reduced manipulator. Examples of two different redundant manipulators are given in Section 4. Finally, Section 5 presents the conclusions.

## 2. Related work

First mentions of redundant joints are found in [34–36]. In these papers a redundant manipulator is designed from a known non-redundant one. In this context, the authors assume that the added joint is the redundant one. In [37] several