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Unified formulation for the stiffness analysis of spatial mechanisms

Alessandro Cammarata

University of Catania, Dipartimento Ingegneria Civile e Architettura, 95125, Viale A. Doria 6, Catania, Italy

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ABSTRACT

This paper presents a complete stiffness analysis of spatial mechanisms. Links flexibility is modeled through structural elements while joints are inherently considered by means of kinematic relations including their degree of freedoms (*dofs*) and degree of constraints (*docs*). Actuation stiffness can be included as well as flexibility of some *docs* can be added in a selective way leaving the remaining rigid. Preloaded joints can be also modeled including joint wrenches. The Condensed Stiffness Matrix (CSM) of an elementary kinematic chain composed of a flexible two-node element and a spatial joint is derived using a robust mathematical formulation based on partitioned matrices and condensation techniques. The CSMs are then combined to find the global stiffness matrix through techniques coming from the structural analysis. The proposed method solves some critical issues of other formulations providing possibility to work with redundant legs (or joints) of fully- and over-constrained PKMs, inherent use of joints without resorting to Lagrangian multipliers, ability to exploit positive-semidefinite joint stiffness matrices without causing a rank-deficient global stiffness matrix; selective inclusion of stiffness in joints along *dofs* and *docs*. Finally, three examples to show the potentiality of the method for different applications of robotics are described.

1. Introduction

The growing interest of the scientific community in robots elasticity is derived from the demand for higher performance. Moving masses reduction is one of the keys to achieve speed and high acceleration such as to shorten the overall working time. However, it must be accompanied by the choice of light and resistant materials and by a careful design to ensure the degree of precision and the ability to transmit or bear loads imposed by the actuators or the environment. Added to these are also attractive applications as soft robotics and compliant robotics that make the elasticity the core of design.

In the early nineties the researchers began to analyze the stiffness of parallel manipulators considering the contribution made by individual legs.

Methods based on Jacobian were the first to address this issue exploiting the duality of kinematics and statics to derive the Cartesian stiffness matrix, that is the matrix mapping the deformations of the end-effector into the wrenches applied upon it.

Gosselin in Ref. [1] used the Jacobian of a Parallel Kinematic Machine (PKM) to define the Cartesian stiffness matrix considering only the stiffness of the actuators. El-Khasawneh and Ferreira in Ref. [2] investigated the problem of finding the minimum

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E-mail address: acamma@diim.unict.it (A. Cammarata).

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and maximum stiffnesses and the directions in which they occur for a parallel manipulator at a given pose. In Refs. [3–6] Zhang and Gosselin proposed the kinetostatic modeling of a family of n-dof PKMs with passive constraining leg in which the degree of freedom (*dof*) of the mechanism is dependent on the passive leg's *dof*. The authors used lumped models for links and joints and modeled link flexibility through virtual joints that allow the motion along the degree of constraint (*doc*) generated by real joints. In 2004 Zhang et al. [7] proposed an extension of the Jacobian matrix method, called Kinetostatic Modelling Method including a complete compliant model developed for the analysis of the PKMs with fixed-length legs. In particular, three types of compliance contribute to deformation of the MP, namely: actuator flexibility, leg bending and axial deformation were included. The main issue of the Kinetostatic Modelling Method seems its limited use to PKMs with fixed length links such as tripods.

The VJM provided by Pashkevich et al. in Ref. [8] is general and can include preloading and loading conditions. In Ref. [9] Pashkevich et al. extended the *Virtual Joint Method* (VJM) to over-constrained parallel manipulators in unloaded and loaded conditions. Even if the VJM is general and can be applied to study loaded equilibrium configurations and detect buckling, some shortcomings are essentially connected to: computational complexity deriving from Jacobians calculation; simplified models to include parallelogram joints or internal loop mechanisms [10].

In Ref. [11] Kim and Lipkin used reciprocal screws to explicitly eliminate the passive joint constraints obtaining leg stiffness matrices that are inherently singular. Starting from the work of Joshi and Tsai [12] and Hong and Choi [13] the concept of Jacobian was extended by means of the screw theory to the *Overall Jacobian* taking into account both the actuation and constraint wrenches imposed upon the MP. In the work of Huang et al. [14] the screw theory and its concepts of reciprocity and duality between twist subspace of permissions/restrictions and wrench subspace of actuation/constraints were used to create a systematic framework based on the Generalized Jacobian. In Ref. [15] the authors applied the Generalized Jacobian to study the stiffness of lower-mobility parallel manipulators. Although these formulations are based on a sound theory, the use of the *observation method* to derive reciprocal screws does not seem suitable for a numerical implementation in the form of algorithm. Alternatively, they require numerical approaches such as *Gram Schmidt algorithm* or *augmentation matrix approach* to obtain the basis of the wrench subspace of constraints necessary to derive the Generalized Jacobian.

Recently, Hoevenaars et al. [16] employed the Generalized Jacobian for the stiffness analysis of lower-mobility parallel manipulators including preloaded compliant joints and deformable links. The method has some limitations when zero stiffness joints are included in the stiffness analysis. Simplificative assumptions must be made to obtain non-square stiffness matrix in joint-space. Besides, only non-redundant kinematic joints in each leg can be taken into consideration.

A different approach from VJM based on the strain energy of deformable systems has been used for the stiffness analysis of PKMs. In Ref. [17] Deblaise et al. exploited the Matrix Structural Analysis (MSA) to perform an analytical stiffness study of PKMs. A clear drawback arises as joint contribute is added by means of Lagrangian multipliers. This issue is overcome in the work of Taghvaeipour et al. [18] where *small-amplitude displacement* (SAD) screw was combined to MSA to obtain the stiffness matrix of mechanical systems in which each joint equation includes a projection operator and elastic coordinates. A similar approach is followed in Ref. [19], while in Ref. [20] the author considered also joints with compliance by introducing *complementary joint-matrix and joint-array* along the *docs* of a joint.

In this paper the methodology developed in Refs. [19–21] is extended to include preload, external wrenches, flexible passive and actuated joints and deformable links inside a unified mathematical formulation based on partitioned matrices and Schur complement, typically used for static condensation in FEM. All aspects are integrated into a unique framework providing rigor and a solid base to the method. In addition, the number of cases is reduced from three to one, making the treatment more robust. Designers and analysts have the opportunity to add joint stiffness and preloading along single *dof* and/or *doc* of the same joint in a selective manner. This offers a variety of solutions that has never been provided before by other stiffness models.

Below, strong points and issues of the proposed method are listed.

1.1. Pros

- Working with redundant legs or joints of fully-constrained or over-constrained PKMs;
- joints are inherently added into the model without resorting to Lagrangian multipliers;
- joint stiffness along *dofs* and/or *docs* can be included in a selective manner and positive-semidefinite joint stiffness matrices can be used;
- parallelogram joint and internal loop mechanisms can be added without simplificative models;
- same nodal partition for elastostatic and elastodynamic analyses.

1.2. Cons

- The Cartesian stiffness matrix cannot be directly derived;
- Loading mode and buckling cannot be analyzed.

The structure of the paper is defined as follows. In Section 2 some key-concepts as small deformation array and joint matrices are recalled. Section 3 is devoted to the derivation of the Condensed Stiffness Matrix including elasticity in joints and links