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Deproximating *Tredgold's Approximation*

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ABSTRACT

Presented is *Tredgold's Approximation* for using an equivalent cylindrical gear with spur teeth to "approximate" a bevel gear with straight teeth. This relation is extended to spiral bevel and hypoid gears by utilizing pitch surface curvature in the direction perpendicular to the gear tooth spiral to establish an equivalent gear. Subsequently, the envelope of a planar gear tooth profile in this perpendicular direction is presented. The envelope of the gear tooth profile is used to determine fully conjugate gear teeth profiles for spatial gear elements. This procedure is valid for any tooth profile along with circular and non-circular gears. To validate the methodology, a virtual model of a bevel gear pair ("*presented model*") is created and an unloaded tooth contact analysis is performed. The procedure used to perform the unloaded tooth contact analysis and determine the corresponding unloaded transmission error is based on the concept of ease-off topography. An example of a face-milled bevel gear pair ("*literature model*") serves as a reference of correctness in determining the ease-off for unloaded contact. To conclude the paper, the obtained ease-off topography and unloaded transmission error of the presented model are calculated and displayed, demonstrating the specification of fully conjugate teeth.

1. Introduction

A gear pair consists of two (wheeled) bodies in direct contact to facilitate general motion transmission between rotating axes. The part of the wheel where the two bodies are in direct contact is known as the gear tooth. The majority of such scenarios involve uniform motion between parallel axes. When this occurs, the bodies are cylindrical and the gear tooth shape is determined by establishing mathematical relations in a plane perpendicular to the gear axes of rotation. This planar surface is known as the transverse plane for cylindrical gears. The term planar gearing is commonly used when referring to such motion transmission due to the planar mathematical relations used to establish the gear tooth shape.

A generalized gear pair can be used to achieve motion transmission between two skew axes. Cylindrically shaped wheeled bodies result when the two rotation axes are parallel. Conically shaped wheeled bodies result when the two rotation axes intersect. Lastly, hyperboloidally shaped wheeled bodies result when the two rotation axes are not parallel nor intersect. These hyperboloidally shaped wheels degenerate into cylinders for motion transmission between parallel axes and cones for motion transmission between intersecting axes. The specification of two surfaces in direct contact that ensure 100% perfect motion

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transmission between skew axes has been a goal of many researchers and no known methodology exists to singly determine the geometry of two such conjugate surfaces.

Gear tooth shape for cylindrically shaped wheeled bodies has been researched for centuries and research in this area continues today, especially with micro-geometry and tooth profile modifications. The premises for determining conjugate gear teeth are to mathematically describe its shape and obtain a relation between the gear tooth shape, the line of force between two gear teeth in direct contact and the desired angular displacement ratio between the two axes of rotation. For planar gearing, such a relation is frequently referred to as Euler's law of gearing. This process is used throughout the world today. To the authors' awareness, no known procedure exists that extends this approach to an applicable procedure to include hyperboloidal gears.

It is viable to specify the shape of one body (say the input gear tooth) and obtain the shape of its "conjugate" mating (output gear tooth) body that ensures desired motion transmission between skew axes. This process is known as the envelope of one body relative to another. In the gearing community, this process is referred to as the equation of meshing. This methodology differs from the above methodology using Euler's law of gearing to establish conjugate gear teeth geometry. Tooth determination via the Euler's law of gearing approach enables the specification of the gear tooth shape (viz., involute, cycloid, and circular-arc) along with path of contact, pressure angle, helix angle, and contact ratio. Knowledge of such features is used to study gears and reduce their costs and size, increase reliability, as well as reduce noise and vibrations. Tooth determination via equation of meshing is the basis in today's face cutting process used to fabricate spiral bevel and hypoid gears. Inherent in this equation of meshing process is the inability to determine conjugate gear teeth independent of each other.

Gear designers have introduced an ease-off function to obtain an appropriate tooth profile modification to achieve conjugate motion for gears produced using a face cutting process. In this case, gear designers iteratively obtain gear tooth geometry by using the equation of meshing and evaluating the loaded tooth contact between the two gears. A limitation of this methodology can be highlighted by recognizing the need to post process such gear elements. Today, it is possible to produce spiral bevel and hypoid gears via contour milling using a 5-axis CNC machine. Interestingly, conjugate gear tooth geometry is typically based on emulating a "successful" face cut gear.

This paper presents the first known methodology that provides gear designers the ability to mathematically specify two gear teeth independent of each other and ensure perfect motion transmission when engaged in mesh. The mathematical model ("presented model") makes use of a special spiral angle on pitch surfaces. The envelope of a planar gear element in a direction perpendicular to the gear tooth spiral is presented to obtain fully conjugate surfaces in direct contact for motion transmission between skew axes. The unloaded transmission error for motion transmission is presented and used in an illustrative example to demonstrate this procedure. The methodology begins by revisiting *Tredgold's Approximation*.

2. *Tredgold's Approximation*

Tredgold's Approximation is presented in many books on gear design for determining the pitch radius and number of teeth that result in an equivalent "bevel" (conical) gear. Implicit is that the pitch radius and number of teeth are for a spur cylindrical gear element. Cylindrical gears are the simplest and most common of the various gear types (viz., bevel, hypoid, worm, wormwheel, and non-circular). Conjugate tooth action for spur cylindrical gears can be easily described and understood using planar geometry. The exact coordinates for a bevel gear's tooth shape along with its varying thickness require an added level of mathematical detail to properly account for its shape and varying tooth thickness. So, it is common to approximate bevel gears with an equivalent spur cylindrical gear to facilitate design. Quantitative expressions are presented to extend this approximation to include spiral bevel and hypoid gear elements. A historical overview of *Tredgold's Approximation* can be found in the Appendix.

Tredgold's Approximation was initiated to describe spur bevel gears with involute teeth. Although originally specified for involute teeth, *Tredgold's Approximation* is applicable to other tooth profiles (viz., cycloid and circular-arc). Radzevich proposed extending *Tredgold's Approximation* to include "crossed axes" gearing by introducing a sphere centered at the intersection between the generator of the pitch surfaces and the common perpendicular to the two axes of rotation [1]. Radzevich used this sphere to generate an equivalent bevel gear pair and did not consider the spiral angle. Apart from *Tredgold's Approximation*, Figliolini and Angeles investigated the path of contact for involute bevel gears on the surface of a sphere and contrasted involute teeth to a crown rack with "flat flanked" teeth [2]. Stachel joined Figliolini and Angeles and together they proposed cycloidal teeth for a certain class of spatial gear elements [3]. Part of this work is rooted in involute gears as suggested by Phillips [4]. More recently, the three redirected their effort to a broader class of gear tooth profiles with line contact (vs point contact) by investigating the instantaneous motion of the common generator between two axodes [5].

Depicted in Fig. 1 is a representation of a bevel gear set in mesh. The outer radii r_p and r_g of the conical axodes are used to define an equivalent spur cylindrical gear set with the same speed ratio. The speed ratio g for a bevel gear set can be defined accordingly:

$$g \equiv \frac{\omega_o}{\omega_i} = \frac{\sin\alpha_p}{\sin\alpha_g} = \frac{N_p}{N_g} = \frac{r_p}{r_g} = \frac{r_{eq\_p}}{r_{eq\_g}}. \tag{1}$$

The equivalent pitch radii r_{eq_p} and r_{eq_g} for the cylindrical gear elements are

$$r_{eq\_p} = \frac{r_p}{\cos\alpha_p} \tag{2}$$