Mechanism and Machine Theory 92 (2015) 144–152

Contents lists available at ScienceDirect

Mechanism and Machine Theory

journal homepage: www.elsevier.com/locate/mechmt

Euler–Rodrigues formula variations, quaternion conjugation and intrinsic connections

Jian S. Dai *

MOE key lab for mechanism theory and equipment design, International Centre for Advanced Mechanisms and Robotics, Tianjin University, PR China
Centre for Robotics Research, School of Natural Sciences and Mathematics, King's College London, University of London, United Kingdom

A R T I C L E I N F O

Article history:
Received 13 May 2014
Received in revised form 16 February 2015
Accepted 7 March 2015
Available online xxxx

Keywords:
Euler–Rodrigues formula
Quaternions
Exponential map
Lie groups
Lie algebras
Kinematics

A B S T R A C T

This paper reviews the Euler–Rodrigues formula in the axis–angle representation of rotations, studies its variations and derivations in different mathematical forms as vectors, quaternions and Lie groups and investigates their intrinsic connections. The Euler–Rodrigues formula in the Taylor series expansion is presented and its use as an exponential map of Lie algebras is discussed particularly with a non-normalized vector. The connection between Euler–Rodrigues parameters and the Euler–Rodrigues formula is then demonstrated through quaternion conjugation and the equivalence between quaternion conjugation and an adjoint action of the Lie group is subsequently presented. The paper provides a rich reference for the Euler–Rodrigues formula, the variations and their connections and for their use in rigid body kinematics, dynamics and computer graphics.

1. Introduction

Euler–Rodrigues formula was first revealed in Euler's equations [1] published in 1775 in the way of change of direction cosines of a unit vector before and after a rotation. This was rediscovered independently by Rodrigues [2] in 1840 with Rodrigues parameters [3] of tangent of half the rotation angle attached with coordinates of the rotation axis, known as Rodrigues vector [4–6] sometimes called the vector–parameter [7], presenting a way for geometrically constructing a rotation matrix. The vector form of this formula was revealed by Gibbs [8], Bisshopp [9], and Bottema and Roth [10] in their presentation of the Rodrigues formulae in planar and spatial motion. In addition to Rodrigues parameters, Euler–Rodrigues parameters were revealed in the same paper [2] as the unit quaternion. It was illustrated by Cayley [11] that the rotation about an axis by an angle could be implemented by a quaternion transformation [12] that was again interpreted by Cayley [13] physically using Euler–Rodrigues parameters that we now know as the quaternion conjugation [14,15], this coincides with the result of using Rodrigues parameters in the Euler–Rodrigues formula, leading to Cayley transform [16] as a mapping between skew–symmetric matrices of Lie algebra elements and special orthogonal matrices of Lie group elements.

The Euler–Rodrigues formula for finite rotations [17,18] raised much interest in the second half of the 20th century. In 1969, Bisshop [9] studied the formula in vector form of the rotation tensor by presenting a derivation from rotating a vector about an axis by an angle. In 1979, Bottema and Roth [10] presented Rodrigues formulae for rigid body displacements of various motions and put forward vectorial representations. In 1980, Gray [4] reviewed Rodrigues' contribution to the combination of two rotations

* Tel.: +442078482321.
E-mail address: jian.dai@kcl.ac.uk.

http://dx.doi.org/10.1016/j.mechmachtheory.2015.03.004

and presented a historical account of this development. In 1989, Cheng and Gupta [19] verified the original contribution of Euler and accounted a further contribution of Rodrigues based on Euler–Rodrigues parameters. Since 1980s, Euler–Rodrigues formula has been widely used in geometric algebra [20,21], theoretical kinematics [10,22,23], and robotics [24]. In modern mathematics, Euler–Rodrigues formula is used as an exponential map [25] that converts Lie algebra $so(3)$ into Lie group $SO(3)$, providing an algorithm for the exponential map without calculating the full matrix exponent [26–28] and for multi-body dynamics [29–32].

In the 21st century, Euler–Rodrigues formula continuously attracted broad interest. In 2003, Bauchau and Trainelli [33] developed an explicit expression of the rotation tensor in terms of vector parameterization based on the Euler–Rodrigues formula and in particular utilized tangent of half the angle of rotations [10,34]. In 2004 and 2006, Dai [3,35] reviewed Euler–Rodrigues parameters in the context of theoretical development of rigid body displacement historically. In 2007, Mebius [36] presented a way of obtaining the Euler–Rodrigues formula by substituting Euler–Rodrigues parameters in a 4×4 rotation matrix based on a quaternion representation. In 2008, Senan and O'Reilly [37] illustrated rotation tensors with a direct product of quaternions and examined the parameter constraint in Euler–Rodrigues parameters. In the same year, Norris [38] applied the Euler–Rodrigues formula to developing rotation of tensors in elasticity by projecting it onto the hexagonal symmetry defined by axes of rotations with Carton decomposition of rotation tensors [39]. In 2010, Müller [40] used a Cayley transformation to obtain a modified vector parameterization that represents an extension of the Rodrigues parameters, which reduces the computational complexity while increasing accuracy. In 2012, Kovács [41] gave a new derivation of the Euler–Rodrigues formula based on a matrix transformation of three continuous rotations. In the same year, Pujol [15,42] investigated the relation between the composition of rotations and the product of quaternions [43] and related the work to Cayley's early contribution [11,44] through Euler–Rodrigues parameters. Following various studies, the use of the Euler–Rodrigues formula and of the Euler–Rodrigues-parameters formulated unit-quaternion has been extended to a broad range of research topics including vector parameterization of rotations [33,40,45], rational motions [46–49], motion generation [50–52] and planning [53], kinematic mapping [54,55], orientation [56] and attitude estimation [57–59], mechanics [60], constraint analysis [61,62], reconfiguration [63,64], mechanism analysis [65–67] and synthesis [68,69], sensing [70] and computer graphics [71–73] and vision [74].

Though various studies were made, reconciliation of different versions of the Euler–Rodrigues formula and their derivations were vaguely known. This paper is to examine all Euler–Rodrigues formula variations, present their derivations and discuss their intrinsic connections to provide readers with a complete picture of variations and connections, leading to understanding of the Euler–Rodrigues formula in its variations and uses as an exponential map and a quaternion operator.

2. Geometrical interpretation of the Euler–Rodrigues formula

Rigid body rotation can be presented in the form of Rodrigues parameters [34,75] that integrate direction cosines of a rotation axis with tangent of half the rotation angle as three quantities in the form of

$$b_x = \tan\frac{1}{2}\theta s_x,\ \ b_y = \tan\frac{1}{2}\theta s_y,\ \ b_z = \tan\frac{1}{2}\theta s_z, \tag{1}$$

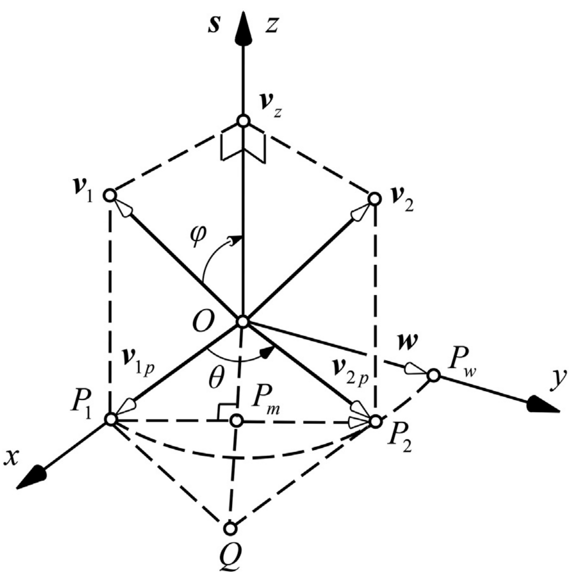

Fig. 1. Projected rhombus and the half-angle of rotation.