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A generalized exponential formula for forward and differential kinematics of open-chain multi-body systems

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ABSTRACT

This paper presents a generalized exponential formula for Forward and Differential Kinematics of open-chain multi-body systems with multi-degree-of-freedom, holonomic and nonholonomic joints. The notion of lower kinematic pair is revisited, and it is shown that the relative configuration manifolds of such joints are indeed Lie groups. Displacement subgroups, which correspond to different types of joints, are categorized accordingly, and it is proven that except for one class of displacement subgroups the exponential map is surjective. *Screw joint parameters* are defined to parameterize the relative configuration manifolds of displacement subgroups using the exponential map of Lie groups. For nonholonomic constraints the admissible screw joint speeds are introduced, and the Jacobian of the open-chain multi-body system is modified accordingly. Computational aspects of the developed formulation for Forward and Differential Kinematics of open-chain multi-body systems are explored by assigning coordinate frames to the initial configuration of the multi-body system, employing the matrix representation of $SE(3)$ and choosing a basis for $se(3)$. Finally, an example of a mobile manipulator mounted on a spacecraft, i.e., a six-degree-of-freedom moving base, elaborates the computational aspects.

1. Introduction

The product of exponentials formula for Forward Kinematics of serial-link multi-body systems with revolute and/or prismatic joints was first introduced by Brockett in 1984 [1]. This formulation was further developed and its roots in Lie group and screw theory were illustrated by Murray et al. in 1994 [2]. One of the most important contributions of this method of multi-body system modeling is the elimination of intermediate coordinate frames in the kinematic analysis of serial-link manipulators. Since then, a number of researchers have investigated the computational efficiency of this formulation [3], and have applied it to different robotic problems [4–8]. In 1995, Park et al. used this formulation to reformulate the dynamical equations of serial-link multi-body systems [9], and later in 2003 Müller et al. attempted to unify the kinematics and dynamics of open-chain multi-body systems with one degree-of-freedom (d.o.f.) joints [10].

The exponential map used in the product of exponential formula is indeed the exponential map of Lie groups, which maps an element of corresponding Lie algebra to an element of the Lie group [11]. For a rigid body, this Lie group is $SE(3)$, which is called the configuration manifold, and the elements of its Lie algebra $se(3)$ are the screws associated with the possible motions of a rigid body in 3-dimensional space [2]. In [12] a family of approximation formulas is presented that allow reconstructing large rigid body motions from a given velocity field, up to a desired order. Screw theory, which was first introduced by Ball in 1900 [13] and also appeared in the work of Clifford [14,15], has been extensively investigated as a powerful means for the kinematic modeling of mechanisms [16–21] and robotic systems [5,22–24], by defining the notion of screw systems [25]. Moreover, the relationship

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Operator

L_r	Left composition/translation by r
R_r	Right composition/translation by r
K_r	Conjugation by r
Ad_r	Adjoint operator corresponding to r
ad_ξ	adjoint operator corresponding to ξ
$[\xi, \eta]$	Lie bracket or matrix commutator
$d_r f$	Differential of the map f at the point r
$T_r M$	Tangent space of the manifold M at the point r
TM	Tangent bundle of the manifold M
$\exp(\xi)$	Group/matrix exponential of ξ
$Lie(G)$	Lie algebra of the Lie group G
$diag(A_1, \ldots, A_n)$	Block diagonal matrix of the entries
$\ltimes$	Semi-direct product of groups
$\|v\|$	Euclidean norm of the vector v
$\tilde{v}$	Skew-symmetric matrix corresponding to the vector v
$R(\theta)$	2×2 rotation matrix for the angle θ
$R(\theta, v)$	3×3 rotation matrix of a rotation for the angle θ, about the vector v

between screw theory, Lie groups and projective geometry in the study of rigid body motion was elaborated in a paper by Stramigioli in 2002 [26]. He subsequently defined the notions of relative configuration manifold and relative screw to study multi-body systems [27]. In 1999 Mladenova also applied Lie group theory to the modeling and control of multi-body systems [28]. As opposed to the geometric nature of most of the above-mentioned works, her approach was mainly algebraic.

Based on a well-known theorem in the theory of Lie groups, any element of a connected Lie group can be written as product of exponentials of some elements of its Lie algebra. Accordingly, Wei and Norman introduced a product of exponentials representation for the elements of a connected Lie group [29], which was adopted by Liu [30] and Leonard et al. [31] to reformulate Kane's equations for multi-body systems and solve nonholonomic control problems on Lie groups, respectively. However, this is not computationally the most efficient way of parameterizing Lie groups, since this parameterization does not use the minimum number of exponentials of the Lie algebra elements (in the product of exponentials). Therefore, in terms of computational efficiency, investigating the surjectivity of the exponential map for Lie groups is valuable. For $SE(3)$, surjectivity of the exponential map is a direct consequence of Chasles' Theorem [2], which implies that any element of $SE(3)$ can be written as the exponential of at least one element of $se(3)$. However, not much work has been done on the exponential parameterization of the Lie subgroups of $SE(3)$. Only for the one-parameter subgroups of $SE(3)$, which correspond to one-d.o.f. joints, the exponential map has been used to parameterize the relative configuration manifold that leads to the standard product of exponentials formula. In fact, it is going to be shown that the Lie subgroups of $SE(3)$ correspond to the relative configuration manifolds of *displacement subgroups* [20,32]. These joints are generally multi-d.o.f. holonomic joints. For generic multi-d.o.f. joints, Stramigioli in [27] briefly mentions that at each point the exponential map can be used as a local diffeomorphism between the relative configuration manifold and its tangent space. He later used this local diffeomorphism to introduce singularity-free dynamic equations of a generic open-chain multi-body system with holonomic and nonholonomic joints [33]. In the following sections, the necessary and sufficient conditions for surjectivity of the exponential map of the relative configuration manifolds of displacement subgroups are given, and under those conditions the corresponding manifolds are parameterized using the elements of their Lie algebras.

In this paper, as a natural extension of the product of exponentials formula, a generalized formulation for Forward and Differential Kinematics of open-chain multi-body systems with multi-d.o.f., holonomic and nonholonomic joints is formalized. Lie group theory and differential geometry are used in Section 2 to classify the multi-d.o.f. joints, and introduce screw joint parameters. In Section 3, exponential map of Lie groups is utilized for parameterization of the relative configuration manifolds of displacement subgroups, and the generalized exponential formula for Forward Kinematics of multi-body systems with displacement subgroups is formally derived. Using the differential of the Forward Kinematics map and an annihilator of the nonholonomic constraints matrix, a coordinate-independent formulation for the Differential Kinematics of an open-chain multi-body system with nonholonomic constraints is derived in Section 4. This formulation is indeed independent of the choice of coordinate chart and a basis for the Lie algebras. Section 5 introduces the computational tools for the utilization of the developed formulation in numerical modeling, and the paper is finalized by a case study in Section 6.

2. Holonomic and nonholonomic joints

A physical 3-dimensional (3D) space can be mathematically modeled as a 3D *affine space*, denoted by A, which is equipped with a vector space V, and a rigid body B is the closure of a bounded open subset of A. Considering a multi-body system $MS(N) =$