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Equal stiffness in all directions: From theory to experiments

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ABSTRACT

This work is concerned with the design of planar frames consisting of rigidly connected beams so that their ending point, which oscillates in the same plane, is characterized by equal stiffness in all directions. Consequently, when being under the action of a static force in the same plane, this point lies on a circle. Its additional property is concerned with equal frequency of vibration not only along the principal axes, but in every direction in the plane. Besides theoretical analytical considerations and a generalization of the approach, this study also provides numerical and experimental analyses and validations.

1. Introduction

One of the main physical parameters of vibrating structures is their stiffness. In order to determine equivalent stiffness characteristics of their corresponding oscillatory mechanical models, one can use three approaches: i) energy methods, ii) force-displacement relationships known from the theory of Strength of Materials, and iii) experimental techniques. Energy methods, in a nutshell, are related to the approach in which a simplified conservative model (for instance, lumped-parameter of reduced number of degrees of freedom) is established such that its kinetic energy and potential energy are equal to that of the original structure [1,2]. The stiffness parameter (spring constant) of certain elastic structural elements can sometimes be determined analytically based on known relationships between force/torque, resulting deformations in axial loading, torsion and bending, as well as Castigliano's theorem [1,3–5]. However, these analytical considerations may sometimes be too cumbersome or even impossible, and the stiffness parameters need to be obtained experimentally [6,7].

In this work, we develop an analytical approach based on Castigliano's theorem imposing a special property for elastic planar frames under consideration: their ending point should have *equal stiffness in all directions*. This property of equal stiffness in all directions is motivated by its practical utilizations, one of which is associated with isochronous systems in which an isotropic elastic property is needed [8]. The design of an original basic model is presented in Section 2, and its modified version for manufacturing requirements, is given in Section 3. The performance characteristics of the former model are validated numerically, and of the latter model experimentally. The generalization of the approach with additional examples is provided in Section 4. The importance of accurate design and manufacture is highlighted through the whole study. Besides the originality of the consideration and the models obtained, the additional advantage achieved is in their potential use on the full range of length scales, from nano-systems to large-scale systems.

2. Basic model and FEM validation

2.1. Theoretical considerations

This section deals with designing geometric parameters of a two-element frame so that its ending Point E (Fig. 1a) has equal stiffness properties in all directions. The frame is assumed to have bending stiffness *EI* (*E* is Young's modulus of elasticity and *I* is the area moment of inertia of the cross-section). Its elements are beams rigidly attached to each other, while the first one is clamped. The aim is to determine the length of the second beam, i.e. the parameter ξ and the angle β between the beams so that the stiffness at Point E is equal in all directions.

The first step of the analysis is to obtain the value of ξ so that the principal axes have the directions labelled by I and II in Fig. 1b [9,10], independent of the value of β. A constant static force **F** acting at Point E is introduced (Fig. 1c). If it acts collinearly with the principal axes, it results in the displacement of Point E only along the axes. These displacements are either maximal or minimal [9]. The extreme displacements of Point E is labelled by δ_{I} when $\varphi = 0$, and by δ_{II} when $\varphi = \pi/2$. These displacements will be calculated based on the strain energy consideration.

The strain energy U is defined by:

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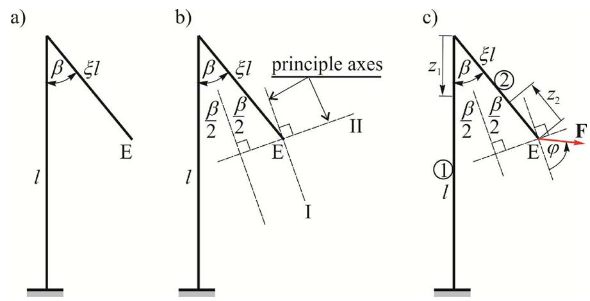

Fig. 1. a) Two-element frame under consideration; b) Two principal axes I and II at Point E; b) Static force **F** acting at Point E.

$$U = \frac{1}{2EI}\left[\int_0^l M_1(z_1)^2 dz_1 + \int_0^{\xi l} M_2(z_2)^2 dz_2\right], \tag{1}$$

where the bending moments expressed in terms of the coordinates z_1 and z_2 (Fig. 1c) along two elements are as follows:

$$M_1(z_1) = -Fz_1 \sin(\varphi + \beta/2) + F\xi l \sin(\varphi - \beta/2), \quad M_2(z_2) = Fz_2 \sin(\varphi - \beta/2). \tag{2}$$

Substituting Eq. (2) into Eq. (1), one has:

$$U = \frac{F^2 l^3}{6EI} f(\varphi, \beta),$$
$$f(\varphi, \beta) = \left[1 - \cos^2\left(\varphi + \frac{\beta}{2}\right) - 3\xi \sin\left(\varphi - \frac{\beta}{2}\right)\sin\left(\varphi + \frac{\beta}{2}\right) + 3\xi^2 - 3\xi^2 \cos^2\left(\varphi - \frac{\beta}{2}\right) + \xi^3 - \xi^3 \cos^2\left(\varphi - \frac{\beta}{2}\right)\right]. \tag{3}$$

The displacement δ in the direction of the force is obtained as $\delta = \partial U/\partial F$:

$$\delta = \frac{Fl^3}{3EI} f(\varphi, \beta). \tag{4}$$

Using the fact that the displacements along the principal axes corresponding to $\varphi = 0$ and $\varphi = \pi/2$ have extreme values, one has:

$$\left.\frac{\partial\delta}{\partial\varphi}\right|_{\varphi=0} = \frac{Fl^3 \sin\beta}{3EI}(1 - 3\xi^2 - \xi^3), \quad \left.\frac{\partial\delta}{\partial\varphi}\right|_{\varphi=\pi/2} = -\frac{Fl^3 \sin\beta}{3EI}(1 - 3\xi^2 - \xi^3). \tag{5}$$

These expressions will be equal mutually when:

$$1 - 3\xi^2 - \xi^3 = 0. \tag{6}$$

This cubic polynomial has only one root that is positive and it is found to be:

$$\xi = 0.53209. \tag{7}$$

This value defines the length of the second beam: its length is about 53% of the length of the first one.

Let us obtain now the angle β. Introducing Eq. (7) and $\varphi = 0$ into Eq. (4), one obtains the displacement along the principal axis I in terms of the angle β:

$$\delta_{\mathrm{I}}(\beta) = 1.19875\frac{Fl^3}{EI}\left(1 - \cos^2\frac{\beta}{2}\right). \tag{8}$$

Substituting Eq. (7) and $\varphi = \pi/2$ into Eq. (4), one derives the displacement along the principal axis II in terms of the angle β:

$$\delta_{\mathrm{II}}(\beta) = 0.13458\frac{Fl^3}{EI}\cos^2\frac{\beta}{2}. \tag{9}$$

Graphical presentations of the displacements given by Eqs. (8) and (9) divided by Fl^3/EI are shown in Fig. 2a for $0 < \beta < \pi$.

It is seen that if the angle β increases, the displacement δ_{I} increases as well, while the displacement δ_{II} decreases. There is a value of β for which these two displacements are equal. The enlarged part of the corresponding region is plotted in Fig. 2b, which implies:

$$\delta_{\mathrm{I}}(\beta) = \delta_{\mathrm{II}}(\beta) \Rightarrow \cos^2\frac{\beta}{2} = 0.89907. \tag{10}$$

The solution of this equation is:

$$\beta = 0.64661 = 37.048°. \tag{11}$$

Substituting $\xi = 0.53209$ and $\beta = 0.64661$ into Eq. (4), one obtains

$$\delta|_{for\ all\ \varphi} = 0.12099\frac{Fl^3}{EI}. \tag{12}$$

Knowing that the corresponding stiffness is $k = F/\delta$, one can finally derive that the stiffness at Point E in all directions is given by:

$$k = 8.2648\frac{EI}{l^3}. \tag{13}$$

The frame whose ending Point E has equal stiffness all directions is shown in Fig. 3a with its characteristic parameters. The equivalent model for the motion of Point E is shown in Fig. 3b, c. In Fig. 3b, the mechanical model for in-plane vibration of Point E is the system with two orthogonal springs of stiffness k given by Eq. (13), but their initial directions correspond to two orthogonal springs coinciding with the principal axes (note that this model with two orthogonal springs coinciding with the principal axes is the only one valid when the stiffness coefficients are different; when the stiffness coefficients are equal mutually, two orthogonal springs can take an arbitrary position [9,10]). Thus, as the stiffness is equal in all directions, any other initial position of these two orthogonal springs is also possible here, as illustrated in Fig. 3c.

Remark. It is interesting to note that the expressions given by Eqs. (8) and (9) enable one to illustrate the extreme displacements along the principal axes, which actually define 'the ellipse of displacement' [9,10] – the displacement of Point E under the action of a static force F. These extreme values are calculated by introducing $C = Fl^3/(EI)$ and presented in Table 1. These values are further illustrated in Fig. 4, which shows how this ellipse of displacement changes with the angle β. It is seen that the ellipse turns into the circle for the value given by Eq. (11), as the extreme displacements are then equal. This figure also points out the necessity to obey the parameters calculated to achieve the behaviour desired – equal stiffness in all directions and the circle as the locus of the displacements under the action of the static force. The ellipses illustrated in Fig. 4b and d confirm that the circle transforms into an ellipse if the angle β is slightly changed from the calculated value.

2.2. FEM analyses

A structure illustrated in Fig. 3a is analysed by FEM (Software Midas NFX 2015 R1) with the parameters: the number of nodes = 116, the number of elements = 115, the number of degrees of freedom = 696. The material was chosen to be 50CrV4 ($E = 200{,}000\,\mathrm{N/mm^2}$, $\rho = 7850\,\mathrm{kg/m^3}$); the length of Element 1 is $l = 150$ mm; the cross-section is circular with the diameter $D = 1$ mm; Eqs. (7) and (11) define the parameters ξ and β. The static force of 0.1 N is applied at Point E in the directions of two principal axes I and II (Fig. 1b) in two cases: when $\beta = 37.048°$ (this is the value given by Eq. (11)) and when the angle does not respect the theoretically derived value, i.e. when $\beta = 45°$. In the former case, the displacements of Point E in the directions of the principal axes are found to be approximately 4.16 mm and equal mutually. Note that Fig. 5a and b has the deformations presented as multiplied so that they are clearly visible, but the emphasis here is on the quantitative measure of the displacement of Point E along the principal axes. Thus, Fig. 5a shows this measure in the direction of the