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Dimension-reduced FPK equation for additive white-noise excited nonlinear structures

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ABSTRACT

The joint probability density function (PDF) of response of a system subjected to Gaussian white noise satisfies the Fokker–Planck–Kolmogorov (FPK) equation, to which neither analytical nor numerical solution is readily available for high-dimensional nonlinear stochastic systems. In the present paper, for the systems excited by additive white noise, by invoking the concept of equivalent drift coefficient, a high-dimensional FPK equation is reduced to a one- or two-dimensional partial differential equation. The equivalent drift coefficient in the new lower-dimensional equation is proved to be the conditional mean function of the drift coefficient in the original high-dimensional FPK equation. The path integral solution is then employed to solve the dimension-reduced FPK-like equation. The response analyses for several systems excited by white noise are exemplified to illustrate the proposed method. The idea proposed in the present paper can be extended to multiplicative white noise and colored noise.

1. Introduction

The response analysis of nonlinear systems excited by stochastic processes is of paramount importance in various areas, in particular in the design of engineering structures subjected to earthquakes or strong wind [1]. For instance, as early as in 1947 Housner suggested to regard seismic ground acceleration process as white noise [2], which was later improved by many researchers, e.g., Kanai [3], Tajimi [4], Hu and Zhou [5], Clough and Penzien [6], leading to various colored noise models described by different power spectral density functions. Nonetheless, white noise excited nonlinear systems still play important roles in stochastic dynamics in that, on the one hand, they are reasonable models for many physical/chemical/biological problems [7], on the other hand, they can be used as the basis for the analysis of non-white noise excited systems, say through the augmentation of dimensions by introducing additional filters [8], or by invoking the stochastic averaging method to convert a colored noise excited system to a Markov system [9,10].

The Fokker–Planck–Kolmogorov (FPK) equation governs the transition and joint probability density function (PDF) of the state vector of a system subjected to white noise excitation. In the past over 50 years, great efforts have been devoted to the solution of FPK equation [11–13]. Important advances have been achieved in both analytical solutions and numerical methods. To the former type belong the analytical solutions for special systems [14,15], and the systematic approach based on the Hamiltonian formulation [10,16], etc. Most analytical approaches aim at stationary solutions. The analytical solution for the transient response, which is of great significance in, e.g., earthquake engineering, however, is much more involved. The numerical methods, on the other hand, can capture both transient and stationary responses. Among others, the finite difference method [17], the finite element method [18], the eigen-function expansion method [19], and the path integral solution (PIS) method [20–23] were extensively studied.

Unfortunately, almost all these analytical and numerical approaches encounter difficulties in high-dimensional nonlinear systems. Till now very few investigations provide results for systems with dimension greater than 10. The underlying difficulty stems essentially from the coupling of nonlinearity and randomness in high-dimensional systems [24]. To reduce the dimension of FPK equation is an alternative approach. For instance, in Er [25] the state vector was firstly split into two sets, one set representing the quantity of interest, the other set to be eliminated by integration. The statistical linearization method was inserted to obtain the joint PDF of the system and then complete the elimination by integration. The accuracy of the approach depends on the degree of nonlinearity because the linearization method was inserted. Besides, in higher-dimensional systems the high-dimensional integration also hinders its applications. To tackle this problem, a probability density evolution method (PDEM) was developed in the past decade [24,26,27],

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which reveals that the change of PDF is driven by the change of state of the underlying physical system. In light of this PDEM, an alternative approach was proposed for dimension reduction of FPK equation [28, 29], where the equivalent flux of probability was reconstructed via the results of PDEM. Further, the concept of equivalent drift coefficient was proposed, resulting in a FPK-like dimension-reduced equation [30].

Along this line, the dimension-reduced FPK equation will be investigated in the present paper. The equivalent drift coefficient of the dimension-reduced equation will be reconstructed as a conditional mean of the drift coefficient of the original high-dimensional FPK equation. Numerical examples are illustrated, demonstrating the effectiveness of the proposed method by comparing the numerical results with the analytical or Monte Carlo simulation results. The paper is organized as follows: in Section 2 the dimension-reduced FPK equation for additive white noise excited systems is derived. The implementation details, including the construction of equivalent drift coefficient and the basic idea of the path integral solution method, are delineated in Section 3. Several numerical examples are studied in Section 4, demonstrating the effectiveness of the proposed method. The concluding remarks are drawn in Section 5.

2. Dimension-reduced FPK equation for nonlinear systems subjected to additive white noise excitation

2.1. The FPK equation

Without loss of generality, the equation of motion of a structure excited by additive white noise reads

$$\mathbf{M}\ddot{\mathbf{X}} + \mathbf{C}\dot{\mathbf{X}} + \mathbf{f}(\mathbf{X}) = \mathbf{L}\boldsymbol{\xi}(t) \tag{1}$$

where $\ddot{\mathbf{X}}, \dot{\mathbf{X}}, \mathbf{X}$ are the n by 1 acceleration, velocity and displacement vectors, respectively; $\mathbf{M}, \mathbf{C}$ are the n by n mass and damping matrices, respectively; $\mathbf{f}(\cdot) = \left(f_1, f_2, \ldots, f_n\right)^{\mathrm{T}}$ is the n by 1 restoring force vector, $\mathbf{L}$ is the n by r loading position matrix, and $\boldsymbol{\xi}(t) = \left(\xi_1, \xi_2, \ldots, \xi_n\right)^{\mathrm{T}}$ is the r by 1 stochastic excitation vectors.

By introducing the state vector $\mathbf{Y} = \left(\mathbf{X}, \dot{\mathbf{X}}\right)^{\mathrm{T}} = (\mathbf{X}, \mathbf{V})^{\mathrm{T}}$, where $\mathbf{V} = \left(V_1, V_2, \ldots, V_n\right)^{\mathrm{T}}$ is adopted to denote the velocity vector to avoid confusion, Eq. (1) is converted to a state equation

$$\dot{\mathbf{Y}}(t) = \mathbf{A}(\mathbf{Y}, t) + \mathbf{B}(\mathbf{Y}, t)\boldsymbol{\xi}(t) \tag{2}$$

or explicitly,

$$\begin{cases} \dot{\mathbf{X}} = \mathbf{V} \\ \dot{\mathbf{V}} = -\mathbf{M}^{-1}\mathbf{C}\mathbf{V} - \mathbf{M}^{-1}\mathbf{f}(\mathbf{X}) + \mathbf{M}^{-1}\mathbf{L}\boldsymbol{\xi}(t) \end{cases} \tag{3}$$

where

$$\begin{cases} \mathbf{A}(\mathbf{Y}, t) = \mathbf{A}(\mathbf{X}, \mathbf{V}, t) = \begin{Bmatrix} \mathbf{V} \\ -\mathbf{M}^{-1}\mathbf{C}\mathbf{V} - \mathbf{M}^{-1}\mathbf{f}(\mathbf{X}) \end{Bmatrix} \\ \mathbf{B}(\mathbf{Y}, t) = \begin{bmatrix} \mathbf{0} \\ \mathbf{M}^{-1}\mathbf{L} \end{bmatrix} \end{cases} \tag{4}$$

In the present paper, only the case when $\boldsymbol{\xi}(t)$ is a Gaussian white noise vector is considered, i.e.,

$$E\left[\boldsymbol{\xi}(t)\right] = \mathbf{0},\ E\left[\boldsymbol{\xi}(t)\boldsymbol{\xi}^{\mathrm{T}}(t')\right] = \mathbf{D}\delta\left(t - t'\right) \tag{5}$$

where $E[\cdot]$ denotes the expectation operator, $\delta(\cdot)$ is Dirac's delta function, $\mathbf{D} = \mathrm{diag}\left(D_1, D_2, \ldots, D_r\right)$ is an r by r diagonal matrix.

Generally, from the physical understanding, Eq. (2) should be understood as an Stratonovich stochastic differential equation (SDE). Further, the Stratonovich SDE could be converted to an equivalent Itô SDE [7], which is mathematically more convenient. Fortunately, because the system in Eq. (2) is additively excited, i.e., $\mathbf{B}(\mathbf{Y}, t)$ does not depend on the state vector $\mathbf{Y}$, as is clear from Eq. (4), the form of the Itô SDE is essentially identical to Eq. (2), which can be further rewritten as

$$d\mathbf{Y}(t) = \mathbf{A}(\mathbf{Y}, t)dt + \mathbf{B}d\mathbf{W}(t), \quad \mathbf{Y}(t_0) = \mathbf{Y}_0 \tag{6}$$

where $\mathbf{W}(t)$ is the r-dimensional Wiener process, which is formally defined by $d\mathbf{W}(t) = \boldsymbol{\xi}(t)\,dt$, and [7]

$$E\left[d\mathbf{W}(t)\right] = \mathbf{0}, \quad E\left[d\mathbf{W}(t)d\mathbf{W}^{\mathrm{T}}(t)\right] = \mathbf{D}dt \tag{7}$$

In Eq. (6) $\mathbf{Y}_0 = \left(\mathbf{X}_0, \mathbf{V}_0\right)$ is the initial value vector, i.e., the initial displacement and initial velocity vector, respectively.

Denote the joint PDF of the state vector by $p_{\mathbf{Y}}(\mathbf{y}, t)$ or $p_{\mathbf{XV}}(\mathbf{x}, \mathbf{v}, t)$, where $\mathbf{y} = \left(y_1, y_2, \ldots, y_{2n}\right)^{\mathrm{T}}$, $\mathbf{x} = \left(x_1, x_2, \ldots, x_n\right)^{\mathrm{T}}$ and $\mathbf{v} = \left(v_1, v_2, \ldots, v_n\right)^{\mathrm{T}}$ denote the real value in the support set. It is well known that the joint PDF of state of the system (6) is governed by the following FPK equation

$$\frac{\partial p_{\mathbf{Y}}(\mathbf{y}, t)}{\partial t} = -\sum_{j=1}^{2n} \frac{\partial\left[A_j(\mathbf{y}, t)p_{\mathbf{Y}}(\mathbf{y}, t)\right]}{\partial y_j} + \frac{1}{2}\sum_{i=1}^{2n}\sum_{j=1}^{2n}\sigma_{ij}\frac{\partial^2 p_{\mathbf{Y}}(\mathbf{y}, t)}{\partial y_i \partial y_j} \tag{8}$$

where the drift coefficient $A_j(\mathbf{y}, t)$ is the jth component of $\mathbf{A}(\mathbf{y}, t)$ in Eq. (4), the diffusion coefficient σ_{ij} is the ijth component of the diffusion matrix specified by

$$\boldsymbol{\sigma} = \mathbf{B}\mathbf{D}\mathbf{B}^{\mathrm{T}} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}^{-1}\mathbf{L}\mathbf{D}\mathbf{L}^{\mathrm{T}}\left(\mathbf{M}^{-1}\right)^{\mathrm{T}} \end{bmatrix} \tag{9}$$

Note here that because $\mathbf{B}$ does not depend on the state vector $\mathbf{y}$, the diffusion matrix $\boldsymbol{\sigma}$ does not depend on the state vector, either.

Generally, the boundary condition of Eq. (8) takes

$$\left.\begin{aligned} p(\mathbf{y}, t)\Big|_{y_j \to \pm\infty} &= 0 \\ y_j p(\mathbf{y}, t)\Big|_{y_j \to \pm\infty} &= 0 \end{aligned}\right\}, \quad j = 1, 2, \ldots, 2n \tag{10}$$

From Eq. (9) it is observed that the component $\sigma_{ij} \neq 0$ holds only for $i = n+1, \ldots, 2n; j = n+1, \ldots, 2n$, then Eq. (8) can be rewritten as

$$\begin{aligned} \frac{\partial p_{\mathbf{XV}}(\mathbf{x}, \mathbf{v}, t)}{\partial t} &= -\sum_{j=1}^{n}\frac{\partial\left[A_j(\mathbf{x}, \mathbf{v}, t)p_{\mathbf{XV}}(\mathbf{x}, \mathbf{v}, t)\right]}{\partial x_j} \\ &\quad - \sum_{j=1}^{n}\frac{\partial\left[A_{j+n}(\mathbf{x}, \mathbf{v}, t)p_{\mathbf{XV}}(\mathbf{x}, \mathbf{v}, t)\right]}{\partial v_j} \\ &\quad + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_{i+n, j+n}\frac{\partial^2 p_{\mathbf{XV}}(\mathbf{x}, \mathbf{v}, t)}{\partial v_i \partial v_j} \end{aligned} \tag{11}$$

As discussed, great efforts have been devoted to the solution of FPK equation, which is still an challenging problem for generic nonlinear systems [12,22]. Some analytical solutions and numerical methods have been extensively studied, providing deep insights into the problems in various areas [31]. However, for the real-world problems with large degrees of freedom and strong nonlinearity, feasible method for the solution of FPK equation is still unavailable.

2.2. Dimension reduction of the FPK equation

As mentioned, the dimension reduction method for the FPK equation is an alternative for high-dimensional systems. Er [25] conducted interesting researches by inserting the equivalent linearization method. Further, in Chen and Yuan [28,29] an equivalent flux based approach was proposed, and was further extended in Chen and Lin [30].

In practice, usually only the PDF of a few quantities of interest, rather than joint PDF of all the state variables, is necessary. Suppose $Y_\ell(t), 1 \leq \ell \leq 2n$, is the quantity of interest. Integrating on both sides of Eq. (8) with respect to the state variables excluding y_ℓ and considering the boundary condition in Eq. (10) yield the following equation

$$\begin{aligned} \frac{\partial p_{Y_\ell}(y_\ell, t)}{\partial t} &= -\frac{\partial \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} A_\ell(\mathbf{y}, t)p_{\mathbf{Y}}(\mathbf{y}, t)dy_1\cdots dy_{\ell-1}dy_{\ell+1}\cdots dy_{2n}}{\partial y_\ell} \\ &\quad + \frac{1}{2}\sigma_{\ell\ell}(t)\frac{\partial^2 p_{Y_\ell}(y_\ell, t)}{\partial y_\ell^2} \end{aligned} \tag{12}$$

where $p_{Y_\ell}(y_\ell, t) = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} p_{\mathbf{Y}}(\mathbf{y}, t)dy_1\cdots dy_{\ell-1}dy_{\ell+1}\cdots dy_{2n}$ is the marginal PDF of $Y_\ell(t)$.

It is noticed that the high-dimensional integral in Eq. (12) yields a function of y_ℓ, which is nothing but the flux of probability in the direction of y_ℓ [28], i.e.,

$$J\left(y_\ell, t\right) = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} A_\ell(\mathbf{y}, t)p_{\mathbf{Y}}(\mathbf{y}, t)dy_1\cdots dy_{\ell-1}dy_{\ell+1}\cdots dy_{2n} \tag{13}$$