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Transient responses of stochastic systems under stationary excitations

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A R T I C L E I N F O

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A B S T R A C T

Stochastic systems to stationary random excitations may fail long before stationarity is achieved. Transient state has to be taken into account. It has been a challenge to obtain exact transient response properties. A novel approximate technique for determining non-stationary probability density function (PDF) of randomly excited nonlinear oscillators is developed. Specifically, it expresses the PDF approximation in terms of polynomial functions with time-dependent coefficients. Based on the results from statistical linearization, residual error of the FPK equation associated with proposed approximation solution can be treated by weighted residual method. As a result, nonlinear ordinary differential equations are produced. Numerical method is adopted to solve these equations and approximate PDF solutions are then obtained. In order to verify the efficiency of the proposed procedure, four examples of stochastic vibrating systems with additional excitations or/and parametric excitations are considered. It is shown that the results obtained by the proposed procedure agree well with those from Monte Carlo simulation.

1. Introduction

Structural systems under random excitations have been investigated for many years. Most work focuses on the stationary excitations or stationary structural responses [1–7], relatively less is related to transient or non-stationary excitations or responses. In this regard, research effort has been in recent years on evolutionary statistics estimation. Note that if the excitation is stationary with arbitrary initial condition (non-stationary start), there will be a transient stage before attaining stationary state. Response in such stage is of importance in estimating structure reliability since failure may occur. One typical example is a structure in an earthquake, in which case structural response during the first several seconds plays a dominant role.

As is well known, when stochastic system is subjected to white noise excitations, the response PDF is governed by Fokker–Planck–Kolmogorov (FPK) equation. If there is a time derivative term in the FPK equation, transient PDF solution can be obtained. However, it is specially difficult to solve it. Exact solutions are available only for linear systems and some special first order nonlinear systems [8,9]. Much research effort is still needed and approximate methods have to be developed. Stochastic linearization has been proved to be one of the most useful approximate techniques for non-linear systems since it does not demand small parameter assumption and extensive computational cost [10–14]. Lately it has been generalized to capture non-Gaussian properties [15], and to determine joint time-frequency nonstationary responses statistics [16]. Furthermore, it can be generalized to moment differential equation method or cumulant-neglect closure method, which enables it to predict higher order moments and reliability estimates with different closure schemes [17–19]. Stochastic averaging method, combined generalized Galerkin method, can be used for nonstationary response envelope probability densities of nonlinear oscillators [20–23]. The main feature of this method is that it enables replacement of the original system by a lower-dimensional one through a combination of time averaging and ensemble averaging. Since this method relies on perturbation, it yields accurate results only for small values of the nonlinearity parameter. Additionally, there is probability density evolution method (PDEM), which is proposed according to the principle of preservation of probability [24]. Due to the difficulty in solving FPK equation, many numerical methods also have been proposed. Monte Carlo simulation (MCS) is the most versatile technique for numerical solutions of stochastic differential equations. However, it associates with numerical convergence, stability, round-off error, and especially requirements for large computational effort in simulating small PDFs in the tail regions. Besides, there are other numerical methods, such as cell mapping method [25], path integration method [26–29], finite element method and finite difference method [30,31], etc.

The exponential-polynomial closure (EPC) method, which is originally proposed for stationary PDF solutions of FPK equation, has been proved to be a useful tool for analyzing nonlinear systems subjected to stochastic excitations [32–35]. A main feature of this method is that it transforms the linear ordinary differential FPK equation to a series of

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nonlinear algebraic equations, which can be solved easily by available numerical methods. Furthermore, the accuracy of approximate solution can be improved by changing the polynomial orders n. In particular, if the interested nonlinear system belongs to stationary potential, the solution obtained by the EPC method coincides with the exact one. More importantly, the EPC method can be extended to multi-degree-of-freedom (MDOF) systems, namely state–space-split EPC (3S-EPC) method [36,37]. Then as a result the investigation on the EPC method can be fundamental for analyzing MDOF systems. Up to now, the EPC method has rarely been used to analyze transient responses of stochastic dynamic systems.

In this paper, the EPC method is improved for approximate transient responses of stochastic systems under Gaussian white noise excitations. Specifically, it expresses the PDF approximation in terms of polynomial functions with time-dependent coefficients. In order to verify the efficiency of the proposed solution procedure, four examples of stochastic vibrating systems with additional excitations or/and parametric excitations are considered. It is shown that the results obtained by using the proposed procedure agree well with those from MCS.

2. Mathematical formulation

2.1. Formulation of the FPK equation

In statistical mechanics and other areas, many problems can be described by the following random systems:

$$\frac{d}{dt}X_i = f_{0i}(\boldsymbol{X},t) + g_{ij}(\boldsymbol{X})W_j(t),$$
$$i = 1,2,\dots,m, \quad j = 1,2,\dots,p, \tag{1}$$

where X_i are components of the vector process $\boldsymbol{X}$, functions $f_{0i}(\boldsymbol{X},t)$ and $g_{ij}(\boldsymbol{X},t)$ are functions determined in the specific case; W_j are zero-mean stationary Gaussian white noise excitations characterized as

$$E[W_i(t_1)W_j(t_2)] = S_{ij}\delta(t_1 - t_2), \tag{2}$$

in which $E[\cdot]$ means stochastic averaging, S_{ij} are constants representing cross-spectral densities of white noises W_i and W_j, $\delta(\cdot)$ is Dirac's delta function. Moreover, if functions $g_{ij}(\boldsymbol{X},t)$ depend only on time t, then stochastic oscillator in Eq. (1) involves with only additive excitations.

The response process $\boldsymbol{X}$ governed by Eq. (1) is a Markov process, which is completely characterized by the transition PDF $p(\boldsymbol{x},t|\boldsymbol{x_0},t_0)$, defined as the PDF of $\boldsymbol{x}$ at time t, subjected to the initial condition $\boldsymbol{x} = \boldsymbol{x_0}$ at $t = t_0$. The transition PDF is ruled by the FPK equation

$$\frac{\partial p}{\partial t} + \frac{\partial p}{\partial x_i}(f_i p) - \frac{1}{2}\frac{\partial^2}{\partial x_i \partial x_j}[G_{ij}(\boldsymbol{x},t)p] = 0, \tag{3}$$

where

$$f_i(\boldsymbol{X},t) = f_{0i}(\boldsymbol{X},t) + \frac{1}{2}S_{ls}\frac{\partial g_{il}(\boldsymbol{X},t)}{\partial X_r}g_{rs}(\boldsymbol{X},t), \tag{4}$$

where the second terms represent so called Wong–Zakai or Stratonovich correction terms. They are vanished in the case of purely external excitations. And $G(\boldsymbol{x},t)$ is

$$G(\boldsymbol{x},t) = S_{ls}g_{il}(\boldsymbol{x},t)g_{js}(\boldsymbol{x},t). \tag{5}$$

The appropriate initial condition associated with the FPK equation is

$$p(\boldsymbol{x},t|\boldsymbol{x_0},t) = \delta(\boldsymbol{x} - \boldsymbol{x_0}). \tag{6}$$

And it is assumed that $p(\boldsymbol{x},t|\boldsymbol{x_0},t)$ fulfills the following constraints

$$p(\boldsymbol{x},t|\boldsymbol{x_0},t) \geq 0$$
$$\lim_{x_i \to \pm\infty} p(\boldsymbol{x},t|\boldsymbol{x_0},t) = 0, \quad i = 1,2,\dots,m$$
$$\int_{R^m} p(\boldsymbol{x},t|\boldsymbol{x_0},t)d\boldsymbol{x} = 1 \tag{7}$$

On the other hand, following Itô's differential rule, moment equations $M_k = x_1^{k_1}x_2^{k_2}...x_n^{k_n}$ can be obtained

$$\frac{d}{dt}E[M_k] = E[f_i(\boldsymbol{x})\frac{\partial M_k}{\partial x_i}] + \frac{1}{2}S_{ls}E[g_{il}(\boldsymbol{x},t)g_{js}(\boldsymbol{x},t)\frac{\partial^2 M_k}{\partial x_i \partial x_j}],$$
$$k = k_1 + k_2 + \cdots + k_n. \tag{8}$$

It follows that response moments of nonlinear systems are governed by an infinite hierarchy of linear differential equations, whose exact solution is impossible. Approximate procedures, commonly called closure schemes, are needed in order to reduce the infinity hierarchy to a finite one. If the closure level is set as $k = 4$, such as cumulant-neglect closure method with truncation level $k = 4$, deviation of moments from Gaussianity can be explicitly described. When the closure scheme $k = 2$, Gaussian moments of different orders at any time instant can be obtained with variance matrix $\sum$ and zero means $\boldsymbol{\mu}$. For clarity's sake, Isserlis's theorem for zero-mean multivariate normal random vector at different time instant is adopted

$$E[x_1 x_2 ... x_{2n};t] = \sum E[x_i x_j;t], \quad i,j = 1,2,\dots,2n.$$
$$E[x_1 x_2 ... x_{2n-1};t] = 0, \tag{9}$$

which are needed in the following solution procedure.

2.2. The developed EPC solution procedure

Since exact solution to Eq. (3) is usually not obtainable, approximate methods have to be adopted. Herein exponential-polynomial closure (EPC) method is used. In order to describe the time evolution of response PDF, time variable t has to be introduced into the previous EPC stationary approximation. Then transient solution is approximated as

$$\tilde{p}(\boldsymbol{x},t) = C\exp[Q_n(\boldsymbol{a},\boldsymbol{x},t)], \tag{10}$$

where C is a normalization constant, $Q_n(\boldsymbol{a},\boldsymbol{x},t)$ are n-degree polynomial functions. Hereto, positivity and normalization constraints in Eq. (7) for PDF solutions are automatically satisfied. In order to guarantee the second constraint in Eq. (7), polynomial order n has to be even to ensure that coefficients of the highest polynomial order term are negative. Polynomial functions $Q_n(\boldsymbol{a},\boldsymbol{x},t)$ are expressed as

$$Q_n(\boldsymbol{a},\boldsymbol{x},t) = \sum_{k=1}^{N_p} a_k(t)x_1^{h_1[k]}x_2^{h_2[k]}...x_m^{h_m[k]}$$
$$h_1[k] = i - j, h_2[k] = j - l, h_3[k] = l - s, \dots .h_n[k] = t;$$
$$i = 1,2,\dots,n; j = 0,1,\dots,i; l = 0,1,\dots,j; s = 0,1,2...l; \dots \tag{11}$$

where N_p is the total number of unknown parameters associated with polynomial orders n, $a_k(t)$ are unknown time-dependent coefficients needed to be specified. Obviously, higher approximation level can be achieved by increasing the number of unknown parameters or the degree of polynomial orders.

Substituting approximate solution Eq. (10) into the FPK equation Eq. (3), residual error is inevitably produced

$$R(\boldsymbol{x},\boldsymbol{a},t) = \frac{\partial \tilde{p}}{\partial t} + f_j\frac{\partial \tilde{p}}{\partial x_j} + \frac{\partial f_j}{\partial x_j}\tilde{p}$$
$$-\frac{1}{2}(\frac{\partial^2 G_{ij}}{\partial x_i \partial x_j}\tilde{p} + \frac{\partial G_{ij}}{\partial x_j}\frac{\partial \tilde{p}}{\partial x_i} + \frac{\partial G_{ij}}{\partial x_i}\frac{\partial \tilde{p}}{\partial x_j} + G_{ij}\frac{\partial^2 \tilde{p}}{\partial x_i \partial x_j}). \tag{12}$$