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A moment-equation-copula-closure method for nonlinear vibrational systems subjected to correlated noise

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ABSTRACT

We develop a moment-equation-copula-closure method for the inexpensive approximation of the steady state statistical structure of strongly nonlinear systems which are subjected to correlated excitations. Our approach relies on the derivation of moment equations that describe the dynamics governing the two-time statistics. These are combined with a non-Gaussian pdf representation for the joint response-excitation statistics, based on copula functions that has (i) single time statistical structure consistent with the analytical solutions of the Fokker–Planck equation, and (ii) two-time statistical structure with Gaussian characteristics. Through the adopted pdf representation, we derive a closure scheme which we formulate in terms of a consistency condition involving the second order statistics of the response, the *closure constraint*. A similar condition, the *dynamics constraint*, is also derived directly through the moment equations. These two constraints are formulated as a low-dimensional minimization problem with respect to the unknown parameters of the representation, the minimization of which imposes an interplay between the dynamics and the adopted closure. The new method allows for the semi-analytical representation of the two-time, non-Gaussian structure of the solution as well as the joint statistical structure of the response-excitation over different time instants. We demonstrate its effectiveness through the application on bistable nonlinear single-degree-of-freedom energy harvesters with mechanical and electromagnetic damping, and we show that the results compare favorably with direct Monte-Carlo simulations.

1. Introduction

In numerous systems in engineering, uncertainty in the dynamics is as important as the known conservation laws. Such an uncertainty can be introduced by external stochastic excitations, e.g. energy harvesters or structural systems subjected to ocean waves, wind excitations, earthquakes, and impact loads [1–6]. For these cases, deterministic models cannot capture or even describe the essential features of the response and to this end, understanding of the system dynamics and optimization of its parameters for the desired performance is a challenging task. On the other hand, a probabilistic perspective can, in principle, provide such information but then the challenge is the numerical treatment of the resulted descriptive equations, which are normally associated with prohibitive computational cost.

The focal point of this work is the development of a semi-analytical method for the inexpensive probabilistic description of nonlinear vibrational systems of low to moderate dimensionality subjected to correlated inputs. Depending on the system dimensionality and its dynamical characteristics, numerous techniques have been developed to quantify the response statistics, i.e. the probability density function (pdf) for the system state. For systems subjected to white noise, Fokker–Planck–Kolmogorov (FPK) equation provides a complete statistical description of the response statistics [7–9]. However, exact analytical solutions of the FPK equation are available only for a small class of systems. An alternative computational approach, the path integral solution (PIS) method, has been developed to provide the response pdf for general nonlinear systems at a specific time instant given the pdf of an earlier time instant. Many studies have been focused on the application of step-by-step PIS method numerically [10–12] and analytically [13–15] reporting its effectiveness on capturing the response statistics. On the other hand, for non-Markovian systems subjected to correlated excitations the joint response-excitation pdf method provides a computational framework for the full statistical solution [16–18]. However, such methodologies rely on the solution of transport equations for the pdf and they are associated with very high computational cost especially when it comes to the optimization of system parameters.

To avoid solving the transport equations for the pdf, semi-

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analytical approximative approaches with significantly reduced computational cost have been developed. Among them the most popular method in the context of structural systems is the statistical linearization method [19–23], which can also handle correlated excitations. The basic concept of this approach is to replace the original nonlinear equation of motion with a linear equation, which can be treated analytically, by minimizing the statistical difference between those two equations. Statistical linearization performs very well for systems with unimodal statistics, i.e. close to Gaussian. However, when the response is essentially nonlinear, e.g. as it is the case for a double-well oscillator, the application of statistical linearization is less straightforward and involves the ad-hoc selection of shape parameters for the response statistics [24].

An alternative class of methods relies on the derivation of moment equations, which describes the evolution of the joint response-excitation statistical moments or (depending on the nature of the stochastic excitation) the response statistical moments [25–27]. The challenge with moment equations arises if the equation of motion of the system contains nonlinear terms in which case we have the well known closure problem. This requires the adoption of closure schemes, which essentially truncate the infinite system of moment equations to a finite one. The simplest approach along this line is the Gaussian closure [28] but nonlinear closure schemes have also been developed (see e.g. [29–37]). In most cases, these nonlinear approaches may offer some improvement compared with the stochastic linearization approach applied to nonlinear systems but the associated computational cost is considerably larger [38]. For strongly nonlinear systems, such as bistable systems, these improvements can be very small. Bistable systems, whose potential functions have bimodal shapes, have become very popular in energy harvesting applications [39–46], where there is a need for fast and reliable calculations that will be able to resolve the underlying nonlinear dynamics in order to provide with optimal parameters of operation (see e.g. [47,48]).

The goal of this work is the development of a closure methodology that can overcome the limitations of traditional closure schemes and can approximate the steady state statistical structure of bistable systems excited by correlated noise. We first formulate the moment equations for the joint pdf of the response and the excitation at two arbitrary time instants [49]. To close the resulted system of moment equations, we formulate a two-time representation of the joint response-excitation pdf using copula functions. We choose the representation so that the single time statistics are consistent in form with the Fokker–Planck–Kolmogorov solution in steady state, while the joint statistical structure between two different time instants is represented with a Gaussian copula density. Based on these two ingredients (dynamical information expressed as moment equations and assumed form of the response statistics), we formulate a minimization problem with respect to the unknown parameters of the pdf representation so that both the moment equations and the closure induced by the representation are optimally satisfied. For the case of unimodal systems, the described approach reproduces the statistical linearization method while for bi-modal systems it still provides meaningful and accurate results with very low computational cost.

The developed approach allows for the inexpensive and accurate approximation of the second order statistics of the system even for oscillators associated with double-well potentials. In addition, it allows for the semi-analytical approximation of the full non-Gaussian joint response-excitation pdf in a post-processing manner. We illustrate the developed approach through nonlinear single-degree-of-freedom energy harvesters with double-well potentials subjected to correlated noise with Pierson–Moskowitz power spectral density. We also consider the case of bi-stable oscillators coupled with electromechanical energy harvesters (one and a half degrees-of-freedom systems), and we demonstrate how the proposed probabilistic framework can be used for performance optimization and parameters selection.

2. Description of the method

In this section, we give a detailed description of the proposed method for the inexpensive computation of the response statistics for dynamical systems subjected to colored noise excitation. The computational approach relies on two basic ingredients:

- *Two-time statistical moment equations*: These equations will be derived directly from the system equation and they will express the dynamics that govern the two-time statistics. For systems excited by white-noise, single time statistics are sufficient to describe the response but for correlated excitation, this is not the case and it is essential to consider higher order moments. Note that higher (than two) order statistical moment equations may be used but in the context of this work two-time statistics would be sufficient.
- *Probability density function (pdf) representation for the joint response-excitation statistics*: This will be a family of probability density functions with embedded statistical properties such as multi-modality, tail decay properties, correlation structure between response and excitation, or others. The joint statistical structure will be represented using copula functions. We will use representations inspired by the analytical solutions of the dynamical system when this is excited by white noise. These representations will reflect features of the Hamiltonian structure of the system and will be used to derive appropriate closure schemes that will be combined with the moment equations.

Based on these two ingredients, we will formulate a minimization problem with respect to the unknown parameters of the pdf representation so that both the moment equations and the closure induced by the representation are optimally satisfied. We will see that for the case of unimodal systems the described approach reproduces the statistical linearization method while for bi-modal systems it still provides meaningful and accurate results with very low computational cost.

For the sake of simplicity, we will present our method through a specific system involving a nonlinear SDOF oscillator with a double well potential. This system has been studied extensively in the context of energy harvesting especially for the case of white noise excitation[41,50–52]. However, for realistic setups it is important to be able to optimize/predict its statistical properties under general (colored) excitation. More specifically we consider a nonlinear harvester of the form

$$\ddot{x} + \lambda\dot{x} + k_1 x + k_3 x^3 = \ddot{y}. \tag{1}$$

where x is the relative displacement between the harvester mass and the base, y is the base excitation representing a stationary stochastic process, λ is normalized (with respect to mass) damping coefficient, and k_1 and k_3 are normalized stiffness coefficients (Fig. 1).

2.1. Two-time moment system

We consider two generic time instants, t and s. The two-time moment equations have been considered previously in [49] for the determination of the solution of a 'half' degree-of-freedom nonlinear oscillator by utilizing a Gaussian closure. We multiply the equation of motion at time t with the response displacement $x(s)$ and apply the mean value operator $\overline{\square}$ (ensemble average). This will give us an equation which contains an unknown term on the right