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A coevolution model for dynamical networks of discrete-time chaotic systems

A. Anzo * **J. G. Barajas-Ramírez** *,[1]

* *IPICyT, Instituto Potosino de Investigación Científica y Tecnológica, División de Matemáticas Aplicadas, Camino a la Presa San José 2055 Lomas 4a Sección CP. 78216, San Luis Potosí, S. L. P., México. (e-mail: andres.anzo@ipicyt.edu.mx; jgbarajas@ipicyt.edu.mx).*

Abstract: Dynamical networks are ensembles of dynamical systems connected with a given structure, which in most cases is assumed to be fixed. However, a more realistic situation is that in which the network topology has its own dynamics. To model a dynamical network with evolving topology is a challenging problem that has recently attracted considerable attention. In this contribution we propose to model the structural evolution of a dynamical network taking into consideration the interplay between the dynamics of the nodes and the processes that determine the structural changes of the network, that is, our model is focus on the coevolution of the dynamical network. In this work we consider a dynamical network with N identical nodes, where each one is a discrete-time chaotic system, namely, the Logistic map. We propose a set of iterative rules that determine the state of each link as either "on" (1) or "off" (0). The state of the link depends on both the states of the nodes which it connects and the structural features of the network. We observed that under node-centric coevolution rules, the network can be made to favor the emergence of a stable synchronized behavior as the structure changes towards a highly connected homogeneous topology. On the other hand, for link-centric coevolution rules, the network can be made to evolve towards a sparsely connected heterogeneous topology, however, the synchronization between its nodes is lost.

Keywords: Complex Networks, Coevolution, Network Synchronization, Chaotic Dynamics.

1. INTRODUCTION

Complex systems can be represented as networks composed by entities called nodes whose interactions are represented by links. The study of networks has focus mainly on the structural aspects of this representation, where nodes and links are assumed to be void of dynamics. In this way, one can use mathematical tools like graph and probability theories to determine key features of the network structure. In particular, if the pattern of connections is not trivial the network is usually called topologically complex, or simply, a complex network [Newman (2010)]. For real-world networks, once the analysis is restricted to static topological aspects, different structural features can be identified, such as the now famous small-world and scale-free effects [Wang and Chen (2003)]. However, its well-known that real-world complex systems are not static, in fact they are in constant evolution. Different processes take place in the system that produce changes in its structure. For example, new nodes and links appear, while others are removed. That is, the network growths or shrinks over time. A complex system can also evolve changing in subtle ways like small variations on the coupling strength of its links. In any case, these change processes constitute the structural evolution of the network.

Dynamical networks are ensembles where each node is a dynamical system. Usually is assume that the connections between nodes are fix, then the dynamics of the entire network are only dependent on the individual dynamics of each node and the static structural features of the network. In recent years, for such static structure dynamical networks, significant results have been obtain in regards to the stability of their collective behavior, e. g. their synchronization [Boccaletti et al. (2006); Wu (2007); Arenas et al. (2008)]. These investigations have made clear that the stability of the synchronized behavior is largely dependent on the structural characteristics of the network. In particular, well-known criteria for synchronization like the Master Stability Function (MSF) [Pecora and Caroll (1998)] and the so-called λ_2 criterion [Li (2005)], have highlighted the crucial influence of the network structure on the node dynamics. Moreover, it is also true that the dynamics of the parts affect the form of the entire system, that is, at the same time that the networks structure affects the dynamics of the nodes, the dynamics of the nodes affect the network structure. There is a co-dependence between these two aspects of real-world complex systems.

In this contribution, we consider the situation in which both the nodes and the network topology have their own dynamics. We call these type of networks, coevolution networks, in allusion to the interplay between the dynam-

[1] Corresponding Author.

ical evolution of the nodes and the processes of structural evolution occurring in the network.

There are works in the literature that look into modeling coevolution, for example in [Sayama et al. (2010)] a model is propose where the local structure of a network in the vicinity of a given node is determine by the dynamics of its neighborhood. In this way, a feedback loop is describe between the dynamical evolution of the nodes and their interconnections. In [Gross and Blasius (2008)] the problem is further investigated coining the term Adaptive Coevolutionary Networks (ACN) to refer to the interaction between the node dynamics and structural change. In particular, ACN as described by Gross and Blasius, are basically generalizations of the Cellular Automata (CA) paradigm, in this model the structure of the network is the result of a CA operated at each node of the network for a given local connectivity describing the vicinity of the CA. An alternative coevolutionary model is propose in [Smith et al. (2009)], there a fixed-size network is connect according to local CA rules describe in terms of structural features of the network, like node degree or centrality measures. In the model proposed by Smith et al. the local rules are iterated over the entire population of nodes, the structure progressively evolves its connection structure over time. In a previous work by the authors [Anzo and Barajas-Ramirez (2011)], a link-based automata description of the structural evolution of a complex network was proposed, in that model a binary CA establishes dynamically the state of the links as either present or absent according to rules dependent on the previous binary state of the link and those of the links in its neighborhood.

In this contribution, we consider a dynamical network with N identical nodes, where each one is a discrete-time chaotic system, namely, the Logistic map. Our coevolution model consists of a set of iterative rules describing the logic that establishes whether the state of a given link is set to either "on" (1) or "off" (0) according to dynamical and structural conditions. We propose two conditions: I) *Dynamical condition.* If the error between the dynamics of the nodes connected by a given link is large, then its state is change, otherwise it is preserve. II) *Topological condition.* If difference between the node degree of the nodes connected by a given link is small, then its state is change, otherwise it remains the same. Using both conditions at the same time, we propose two distinct types of coevolution rules: *Node-centric* coevolution rule, which give preference to the dynamical conditions over structural ones, and *Link-centric* coevolution rule, which in contrast, gives preference to structural considerations over dynamical ones.

Our results show that under node-centric coevolution rules, the network can be made to favor the emergence of a stable synchronized behavior, while the structure evolves towards a highly connected homogeneous topology. However, if we use link-centric coevolution rules, the network can be made to evolve towards a sparsely connected heterogeneous topology, however, the synchronization between the nodes is lost.

The remainder of the paper is organized as follows: In Section 2, our coevolution model is described with detail. In Section 3, we provide some numerical results for both node-centric and link-centric coevolution rules. In Section 4, we provide some closing remarks.

2. COEVOLUTION MODEL

Consider a network of N identical discrete-time dynamical systems coupled by linear, unweighted, and bidirectional links with their own dynamics. The state equation of the entire network is

$$x_i^{k+1} = f(x_i^k) + c\sum_{j=1}^{N} a_{ij}^k f(x_j^k), \quad i = 1, \ldots, N. \tag{1}$$

where $x_i^k \in \Re$ is the state variables of the i-th node at the discrete time k. The map $f : \Re \rightarrow \Re$ describes the dynamics of a single isolated node, we consider that each node is a chaotic Logistic map given by

$$f(x^k) = rx^k(1 - x^k) \tag{2}$$

with $r = 3.9$. The coupling for each node is given by the uniform coupling strength $c \in \Re$, and the time dependent coupling matrix $\mathcal{A}^k = \{a_{ij}^k\} \in \Re^{N\times N}$, which describes the current state of the network topology as follows: If at the time k the entry $a_{ij}^k = a_{ji}^k = 1$, then the link between the node i-th and j-th node is active, that is, its state is set to "on". On the other hand, if the entry $a_{ij}^k = a_{ji}^k = 0$, then the link is inactive with its state set to "off". The diagonal entries are given by the following equation:

$$a_{ii}^k = -\sum_{j=1}^{N} a_{ij}^k = -\sum_{i=1}^{N} a_{ij}^k = -d_i^k \tag{3}$$

where d_i^k is the node degree of the i-th node at time k, such that at all times the connectivity is diffusive.

In order to establish the state of a given link we use the following conditions:

I. **Dynamical condition (δ_{ij}^k).**
If the error between the i-th and j-th nodes is small ($\epsilon_{ij}^k = |x_i^k - x_j^k| < 0.2$), we say that the condition for change is not satisfy ($\delta_{ij}^k = 0$), otherwise, the dynamical condition for change is present ($\delta_{ij}^k = 1$).

II. **Structural condition (σ_{ij}^k).**
If the difference between the node degree of the i-th and j-th nodes is small ($\mu_{ij}^k = |d_i^k - d_j^k| < 2$), we say that the structural condition for change is present ($\sigma_{ij}^k = 1$), otherwise, the structural condition for change is not satisfy ($\sigma_{ij}^k = 0$).

Using these conditions we define two distinct coevolution rules:

A. **Node-centric coevolution rule**.
If the dynamical condition is not present ($\delta_{ij}^k = 0$), the state of the link remains without change, regardless of the structural condition. However, if the dynamical condition for change is present ($\delta_{ij}^k = 1$), we take into consideration the structural condition and the current state of the link. Then, the new state of the link is determine according to the logic in Table 1.

B. **Link-centric coevolution rule**.
If the structural condition is not present ($\sigma_{ij}^k = 0$), the state of the link remains without change, regardless of the dynamical condition. On the other hand, if the structural condition for change is present ($\sigma_{ij}^k = 1$),