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Positive trigonometric polynomials for strong stability of difference equations

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Abstract: We follow a polynomial approach to analyse strong stability of linear difference equations with several independent delays. Upon application of the Hermite stability criterion on the discrete-time homogeneous characteristic polynomial, assessing strong stability amounts to deciding positive definiteness of a multivariate trigonometric polynomial matrix. This latter problem is addressed with a converging hierarchy of linear matrix inequalities (LMIs). Numerical experiments indicate that certificates of strong stability can be obtained at a reasonable computational cost for state dimension and number of delays not exceeding 4 or 5.

Keywords: strong stability, spectral radius, trigonometric polynomials, LMI

1. INTRODUCTION

In general, spectrum-based analysis of time-delay systems can be handled in the same way it is done for delay-free systems. Although the spectrum is infinite, stability is determined by the rightmost eigenvalues, more precisely by the sign of the spectral abscissa, the maximum real part of the eigenvalues. For retarded systems, the spectral abscissa is nonsmooth but continuous in all parameters of the system, including time delays, see Vanbiervliet et al. (2007). However, it results from Henry (1974); Avellar and Hale (1980); Hale and Verduyn Lunel (1993, 2002), that, in general, it is not the case for neutral systems and kernel operators - the so-called associated difference equation, see also Michiels et al. (2002); Michiels and Vyhlídal (2005); Michiels and Niculescu, (2007). It is well-known that the spectral abscissa of the difference equation is not continuous in delays. Thus, arbitrarily small changes in the delay values can destroy stability. Moreover, it can even happen that the number of unstable roots increases stepwise from zero to infinity. In order to handle this hypersensitivity of the stability with respect to delay values, the concept of *strong stability* was introduced by Hale and Verduyn Lunel (2002) for delay difference equations. Let us remark that the *strong stability* concept has recently been generalized by Michiels, et al., (2009) toward difference equations with dependencies in the delays.

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As stability of its kernel operator is a necessary condition for stability of a neutral system, all the hypersensitivity stability issues are carried over to the stability of neutral systems. Thus the *strong stability* test should always be performed to guarantee practical stability of neutral systems. However, as will be shown later in the text, the *strong stability* test is rather complex. So far, a coarse numerical implementation of the test without guarantee or certificate has been used as a rule, see e.g. Michiels and Vyhlídal (2005); Vyhlídal et al. (2010). Even though this *brute force* based approach works in most cases, it might fail due to approximation errors in the numerical scheme. As the main result of this paper we propose a more rigorous *strong stability test* that is based on a polynomial approach, relying on the numerical solution of a hierarchy of linear matrix inequalities (LMIs).

In the field of time-delay systems, LMIs are usually used as stability determining criteria resulting from the Lyapunov time-domain approach, see e.g. Niculescu (2001) or Li et al. (2008), among many others.

1.1 Problem statement

We consider a neutral system of the following form

$$\frac{d}{dt}\left(x(t)+\sum_{k=1}^{m}H_k x(t-\tau_k)\right)=A_0\,x(t)+\sum_{j=1}^{p}A_j x(t-\vartheta_j) \tag{1}$$

where $x\in\mathbb{R}^n$ is the state, $\tau_k>0, k=1,\ldots m$ and $\vartheta_j>0, j=1,\ldots p$ are the time delays. It is well-known, see Hale and Verduyn Lunel (1993), that a necessary

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condition for stability of neutral system (1) is stability of the associated difference equation

$$x(t) + \sum_{k=1}^{m} H_k x(t - \tau_k) = 0. \qquad (2)$$

Moreover, *strong stability* of equation (2) is required, i.e. stability independent of the values of the delays, Avellar and Hale (1980); Hale and Verduyn Lunel (1993). In Hale and Verduyn Lunel (2002) (Theorem 2.2 and Corollary 2.2), a condition for strong stability is stated as follows:

Proposition 1. Delay difference equation (2) is strongly stable if and only if

$$\gamma_0 := \max_{\theta \in [0,\ 2\pi]^m} r_\sigma \left(\sum_{k=1}^{m} H_k e^{-i\theta_k} \right) < 1, \qquad (3)$$

where r_σ denotes the spectral radius, i.e. the maximum modulus of the eigenvalues.

Notice that the quantity γ_0 does not depend on the value of the delays, i.e. exponential stability locally in the delays is equivalent with exponential stability globally in the delays Hale and Verduyn Lunel (2002).

Let us remark that by homogeneity, the expression of γ_0 can be simplified to

$$\gamma_0 = \max_{\theta \in [0,\ 2\pi]^{m-1}} r_\sigma \left(\sum_{k=1}^{m-1} H_k e^{-i\theta_k} + H_m \right). \qquad (4)$$

We conclude the section with some properties of the quantity γ_0, see Michiels and Niculescu, (2007); Michiels and Vyhlídal (2005), for more details.

Properties

(1) Stability of difference equation (2) with rationally independent [1] delays implies strong stability, and vice versa
(2) In the case of one delay ($m = 1$),
$$\gamma_0 = r_\sigma(H_1).$$
(3) In the case of a scalar equation ($n = 1$),
$$\gamma_0 = \sum_{k=1}^{m} |H_k|.$$
(4) A sufficient, but as a rule conservative, condition for strong stability is given by
$$\sum_{k=1}^{m} \|H_k\| < 1$$
where $\|.\|$ denotes the matrix Euclidean norm, i.e. the maximum singular value.

1.2 Computational issues

The problem of solving (3) can be formulated as an optimization task with the objective to find the global maximum of spectral radius over $\theta \in [0,\ 2\pi]^m$. However, in general the objective function $r_\sigma(\theta)$ is nonconvex, i.e. it can have multiple local maxima. Besides, the function can be nonsmooth (e.g. at the points where the spectral radius is determined by more than either one single eigenvalue or a couple of complex conjugate eigenvalues). The fact that the function is nonsmooth precludes the use of standard optimization procedures. Instead, nonsmooth optimization methods can be used, such as gradient sampling, see Burke et al. (2005); Overton (2009). However, even though these methods can handle the problem of nonsmoothness, they converge to local extrema as a rule. As suboptimal solutions are not sufficient (the global maximum of the spectral radius is needed) a brute force method has been used to solve the task so far, see Michiels and Vyhlídal (2005); Michiels, et al., (2009); Vyhlídal et al. (2010). In the first step, each dimension of $[0,\ 2\pi]^m$ is discretized to N points. Then evaluation of (3) consists in solving N^m times $n \times n$ eigenvalue problems. Hence, the overall cost of one evaluation of γ_0 is $O\left(N^m n^3\right)$, see Vyhlídal et al. (2010). If the simplified expression (4) is used, the computational costs reduces to $O\left(N^{m-1} n^3\right)$. Obviously, the complexity of the computation grows considerably with the number of delays in the difference equation. Moreover, the risk of missing global extrema due to sparse or inappropriate gridding cannot be avoided.

[1] The m numbers $\tau = (\tau_1, \ldots, \tau_m)$ are rationally independent if and only if $\sum_{k=1}^{m} n_k \tau_k = 0,\ n_k \in \mathbb{Z}$ implies $n_k = 0,\ \forall k = 1, \ldots, m$. For instance, two delays τ_1 and τ_2 are rationally independent if their ratio is an irrational number.

2. STRONG STABILITY AND HERMITE'S CONDITION

Consider the characteristic polynomial

$$p(z) = \det(z_0 I_n + \sum_{k=1}^{m} z_k H_k), \qquad (5)$$

which is homogeneous of degree n in $m + 1$ variables z_k, $k = 0, 1, \ldots, m$.

Based on (3), considering $z_k = e^{j\theta_k}, \theta_k \in [0, 2\pi], k = 1, .., m$, the difference equation (2) is strongly stable if and only if the univariate polynomial

$$z_0 \to p(z)$$

is discrete-time stable,i.e. it has all its roots in the open unit disk.

In order to deal with stability of this polynomial, we use a stability criterion based on the Hermite matrix. It is a Hermitian matrix of dimension n whose entries are quadratic in the coefficients of the polynomial. The Hermite matrix $z_1, \ldots, z_m \to H(z)$ is therefore a trigonometric polynomial matrix in m variables $z_1, \ldots, z_m$.

Derived by the French mathematician Charles Hermite in 1854, the Hermite matrix criterion is a symmetric version of the Routh-Hurwitz criterion for assessing stability of a polynomial. It says that a polynomial $p(z) = p_0 + p_1 z + \cdots + p_n z^n$ has all its roots in the open upper half of the complex plane if and only if its Hermite matrix $H(p)$ is positive definite. Note that $H(p)$ is n-by-n, Hermitian and quadratic in coefficients p_k, so that the above necessary and sufficient stability condition is a quadratic matrix inequality (QMI) in coefficient vector $p = [p_0\, p_1\ \cdots\ p_n]$.

The standard construction of the Hermite matrix goes through the notion of Bézoutian, a particular form of the resultant. A bivariate polynomial is constructed,